On better-quasi-ordering under graph minors

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On blackboard

Conjecture (Folklore)

The finite graphs are better-quasi-ordered (BQO) under the minor relation.

Conjecture (Thomas '88)

The countable graphs are well-quasi-ordered (WQO) under the minor relation.

WQO vs. BQO

“There is not much chance of proving these conjectures because they imply that the set of all finite graphs is **‘second-level better-quasi-ordered’** *by minor containment, which in itself seems to be a hopelessly difficult problem”.*

— Robertson, Seymour & Thomas ’95

2nd level BQO

Given two sets of graphs $\mathcal{G}, \mathcal{G}'$, we write $\mathcal{G} <_* \mathcal{G}'$ if for every $G \in \mathcal{G}$ there is $H \in \mathcal{G}'$ such that $G < H$.

Problem

Are the sets of finite graphs WQO under $<_*$?

Equivalently:

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Is the set of minor-closed families of finite graphs WQO under $\subseteq$?

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Yes if Conjecure 1 is true.

Better-Quasi-Ordering (BQO)

Given two sets of graphs $\mathcal{G}, \mathcal{G}'$, we write $\mathcal{G} <_* \mathcal{G}'$ if for every $G \in \mathcal{G}$ there is $H \in \mathcal{G}'$ such that $G < H$.

Note that $<_*$ is a quasi-order on $\mathcal{P}(\mathcal{F})$, where $\mathcal{F} := \{$ Finite graphs $\}$.

We can iterate:

If $\mathcal{P}(\mathcal{F})$, $\mathcal{P}(\mathcal{P}(\mathcal{F}))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{F})))$, ... are WQO for ω_1 steps, this defines what it means for $<$ to be a BQO.

BQO Theorems

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The countable linear-order types are BQO (hence WQO) under the embeddability relation.

Theorem (Thomassé '00)

... Same for Countable Series-Parallel Orders.

Theorem (Martinez-Ranero '11)

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... Same for closed subsets of zero-dimensional Polish spaces.

Theorem (Thomas '89)

... Same for graphs of tree-width k.

WQO vs. BQO

"the poset of finite graphs endowed with the minor relation is the only naturally occurring WQO which is not yet known to be BQO".

— Y. Pequignot '17

The Theorem

A graph is rayless if it does not contain an (1-way) infinite path (aka. ray).

Theorem (G '25+)

The following statements are equivalent:

1. *The finite graphs are BQO;*
2. *The countable rayless graphs are WQO;*
3. *The countable rayless graphs are BQO.*

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Theorem (R. Schmidt '83)

A (countable) graph G is rayless iff $G \in Rank_\alpha$ for some ordinal α ($< \omega_1$).

We write $Rank(G) = \alpha$ for the smallest such α.

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Can we prove Thomas' conjecture for rayless graphs by induction on the rank? Beyond reach...

Theorem 2

Let $\mathcal{R}$ denote the class of countable rayless graphs.

Theorem (G '25+)

The following statements are equivalent:

1. *$\mathcal{R}$ is WQO;*
2. *$\mathcal{R}$ has no infinite descending chain;*
3. *$\mathcal{R}$ has no infinite antichain;*
4. *for every ordinal $\alpha < \omega_1$, the number of minor-twin classes of countable rayless graphs of rank α is $\aleph_0$.*

Say that G, G' are minor-twins, if $G < G'$ and $G' < G$.

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A rank-reducing lemma

Let $UF := \{G \mid$ each component of G is finite $\}$.

Corollary

The number of minor-twin classes of UF is $\aleph_0$
<=> the finite graphs are WQO (=GMT).

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For a rayless G, let
$\mathcal{C}(G) := \{ H \mid H < G$ and $Rank(H) < Rank(G) \}$
(For $G \in UF$, these are the finite minors of G.)

Lemma

For every $G, G' \in UF$, *we have*
$\mathcal{C}(G) \subseteq \mathcal{C}(G')$ <=> $G < G'$.

Corollary 2: Self-minors

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Conjecture (Seymour's self-minor conjecture (unpublished))

Every (countable) graph is a proper minor of itself.

False for uncountable graphs [Oporowski '90].

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True for rayless graphs (of any cardinality).

Corollary 2: Self-minors

Lemma

For every $G, G' \in \mathit{Rank}_1$, we have $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(G') \iff G < G'$.

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Corollary 2: Self-minors

Lemma

For every $G, G' \in$ $Rank_1$, we have $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(G')$ <=> $G < G'$.

where $\mathcal{C}^\bullet(G) := \{$ finite (H, M) $|$ $(H, M) <_\bullet (G, A(G))$ and $Rank(H) < Rank(G)$ $\}$

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The Rank-reducing lemma

Lemma

For every $G, G' \in$ Rank$_1$, we have $C^\bullet(G) \subseteq C^\bullet(G') <=> G < G'$.

More generally:

Lemma

Let $G, H \in$ Rank$_\alpha$, $\alpha < \omega_1$. Assume $\mathrm{Rank}^\bullet_{<\alpha}$ is WQO, and $|\mathrm{Rank}^\bullet_{<\alpha}|_{<_\bullet}$ is countable. We have $G < G' <=> C^\bullet(G) \subseteq C^\bullet(G')$.

Theorem 2 again

$\mathcal{R}$:= { countable rayless graphs }.

Theorem (G '25+)

The following statements are equivalent:

1. $\mathcal{R}$ *is WQO; (<=>* $\mathcal{F}$ *is BQO <=>* $\mathcal{R}$ *is BQO)*
2. $\mathcal{R}$ *has no infinite descending chain;*
3. $\mathcal{R}$ *has no infinite antichain;*
4. *for every ordinal* $\alpha < \omega_1$*, the number of minor-twin classes of countable rayless graphs of rank* α *is* $\aleph_0$*.*

Theorem 2 again

Theorem (G '25+)

The following are equivalent for every $\alpha < \omega_1$:

1. $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *is WQO for every* $n \in \mathbb{N}$;
2. $|\mathrm{Rank}_\alpha|_< = \aleph_0$;
3. $|\mathrm{Rank}^\bullet_\alpha|_{<_\bullet} = \aleph_0$;
4. $|\mathrm{Rank}_\alpha|_< < 2^{\aleph_0}$;
5. $\mathrm{Rank}^\bullet_\alpha$ *has no descending chain;*
6. Rank_α *has no descending chain;*
7. $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *has no antichain for every* $n \in \mathbb{N}$;

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7. $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *has no antichain for every $n \in \mathbb{N}$;*

Lemma

Let $G, H \in Rank_\alpha, \alpha < \omega_1$. Assume $\mathrm{Rank}^\bullet_{<\alpha}$ is WQO, and $|\mathrm{Rank}^\bullet_{<\alpha}|_{<_\bullet}$ is countable. We have $G < G' <=> \mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(G')$.

Theorem 1 again

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Theorem (Folklore; Pequignot '17)

A quasi-order $\mathcal{F}$ is BQO if and only if HCIP($\mathcal{F}$) is WQO.

where *HCIP($\mathcal{F}$)* denotes the set of hereditarily countable elements of $\mathcal{F} \cup \mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{P}(\mathcal{F})) \cup \ldots$ (up to ω_1).

Forbidding a rayless tree

Theorem (G '25+)

Let $\mathcal{T}$ be a minor-closed family of $\mathbb{N}$-labelled rayless forests. Then $\mathcal{T}$ is **Borel** *if and only if it is proper, i.e. it does not contain all rayless forests.*

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Proof ingredients:

— Thomas' theorem that the graphs of tree-width k are WQO;

— The rank-reducing lemma;

— Hacking the Turing machine (with J. Grebik).

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Problem

Is there a family of rayless $\mathbb{N}$-labelled graphs which is closed under minors, has rank less than ω_1, and is not Borel?

If yes then the finite graphs are not BQO.

Subclasses

Problem

Are the finite **planar** *graphs BQO?*

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