Infinite Cycles in Graphs

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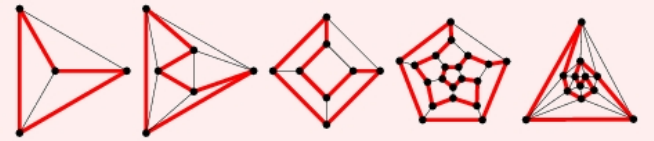
Hamilton cycles

Hamilton cycle: A cycle containing all vertices.

Some examples:



Tutte's Theorem

Theorem (Tutte '56)

Every finite 4-connected planar graph has a Hamilton cycle

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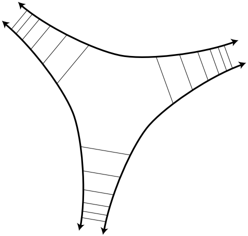
Every finite 4-connected planar graph has a Hamilton cycle

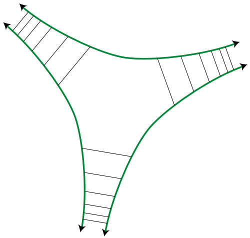
Theorem (Yu '05)

Every locally finite 4-connected planar graph with at most 2 ends has a spanning double ray

Infinite Cycles



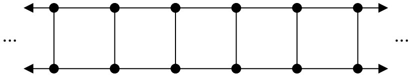
Infinite Cycles



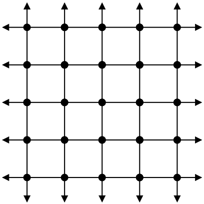
?

What is an infinite cycle?

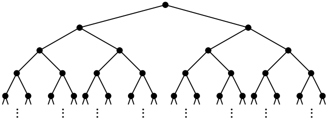
Ends



two ends



one end

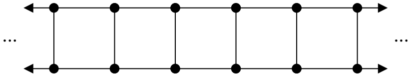


2^ω many ends

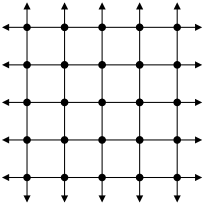
Ends

end: equivalence class of rays

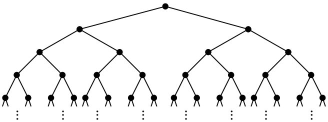
two rays are equivalent if no finite vertex set separates them



two ends

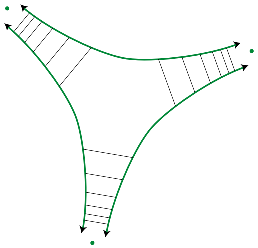


one end



2^ω many ends

The Freudenthal compactification



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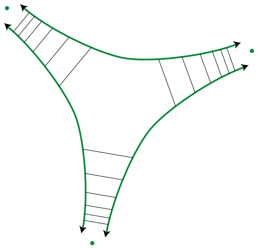

Circle:
A homeomorphic image of S^1 in $|\Gamma|$.

Infinite cycles

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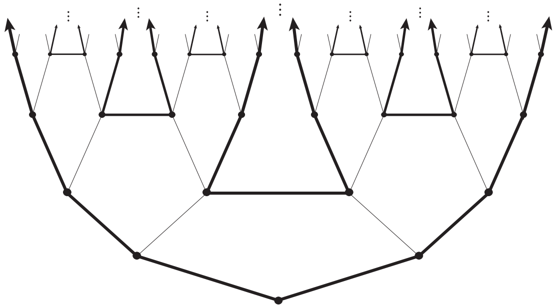

Infinite cycles

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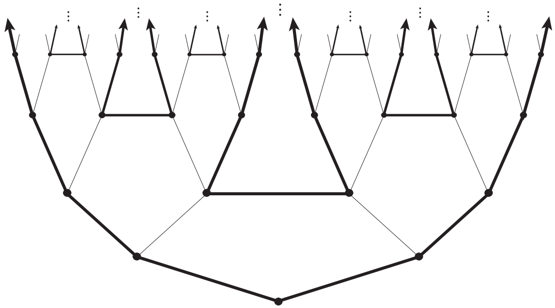

Hamilton circle:
a circle containing all vertices

Infinite cycles

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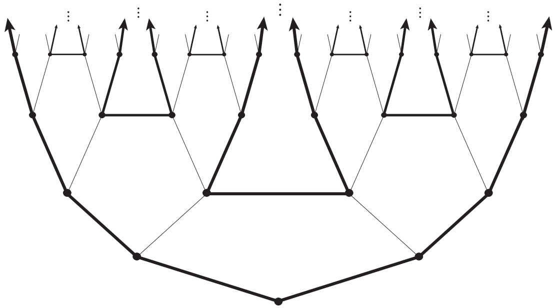
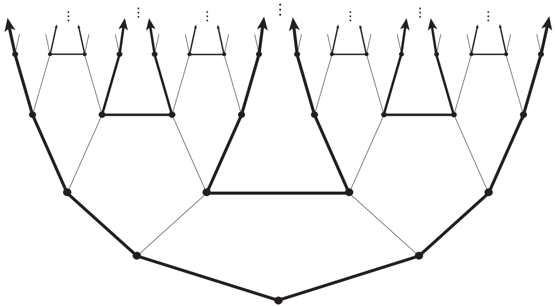


Hamilton circle:
a circle containing all vertices (and all ends?)

Infinite cycles

Circle:
A homeomorphic image of S^1 in $|\Gamma|$.



Hamilton circle:
a circle containing all vertices (and thus also all ends).

Fleischner's Theorem

Theorem (Fleischner '74)

The square of a finite 2-connected graph has a Hamilton cycle

Fleischner's Theorem

Theorem (Fleischner '74)

The square of a finite 2-connected graph has a Hamilton cycle

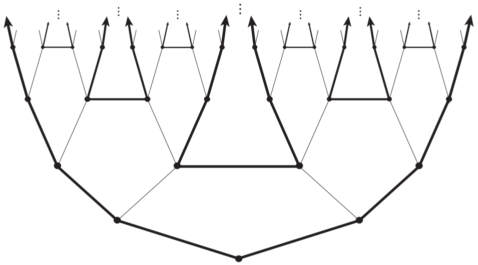
Theorem (Thomassen '78)

The square of a locally finite 2-connected 1-ended graph has a spanning double ray.

Fleischner's Theorem for Locally Finite Graphs

Theorem (G '06)

The square of a locally finite 2-connected graph has a Hamilton circle



Hamiltonicity in Groups

Problem (Lovasz '69)

Does every finite Cayley graph have a Hamilton cycle?

Hamiltonicity in Groups

Problem (Lovasz '69)

Does every finite Cayley graph have a Hamilton cycle?

Problem

Does every 1*-ended Cayley graph have a Hamilton circle (i.e. a spanning double ray)?*

Hamiltonicity in Groups

Problem

Prove that a Cayley graph with infinitely many ends has a Hamilton circle iff it has property A.

Hamiltonicity in Groups

Problem

Prove that a Cayley graph with infinitely many ends has a Hamilton circle iff it has property A.

Problem

Define property A so that the assertion above becomes true.

Connectednes vs. path-connectedness

Problem (Diestel)

Is every (topologically) connected subspace of $|\Gamma|$ *path-connected?*

Connectednes vs. path-connectedness

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Is every (topologically) connected subspace of $|\Gamma|$ path-connected?

Theorem (G '04)

No.

Connectednes vs. path-connectedness

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Is every (topologically) connected subspace of $|\Gamma|$ path-connected?

Theorem (G '04)

No.

Corollary

Connectedness does not imply path-connectedness in the hypebolic compactification of a hyperbolic Cayley graph.

The Cycle Space of a Finite Graph

$\mathcal{C}(\Gamma)$

- A vector space over $\mathbb{Z}_2$
- Consists of sums of circuits

The Cycle Space of an Infinite Graph

Known facts:

- A connected graph has an Euler tour iff every edge-cut is even
- G is planar iff $\mathcal{C}(\Gamma)$ has a simple generating set
- If G is 3-connected then its peripheral circuits generate $\mathcal{C}(\Gamma)$
- The geodetic cycles generate $\mathcal{C}(\Gamma)$

Generalisations:

Bruhn & Stein

Bruhn

Bruhn & Stein

G & Sprüssel

Failure in “continuous” problems

Theorem (G & Sprüssel)

The geodetic circles of a locally finite graph Γ *generate* $\mathcal{C}(\Gamma)$

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Similarly:

Theorem (G)

In a locally finite electrical network the circles also satisfy Kirchhoff's 2nd law if the lengths (i.e. the resistances) of the edges respect $|\Gamma|$.

$\ell - TOP$

Assign lengths $\ell : E(\Gamma) \to \mathbb{R}^+$

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Theorem (G)

If $\sum_{e \in E(\Gamma)} \ell(e) < \infty$ then $\ell - TOP(\Gamma) = |\Gamma|$.

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Assign lengths $\ell : E(\Gamma) \to \mathbb{R}^+$

$\longrightarrow$ distance function d_ℓ

Let $\ell - TOP(\Gamma)$ be the completion of (Γ, d_ℓ).

Theorem (G)

In a locally finite electrical network the circles in $\ell - TOP$ satisfy Kirchhoff's 2nd law.

Your ‘To Do’ List

- read Diestel’s expository paper
  The cycle space of an infinite graph
  or Chapter 8.5. of Diestel’s book;

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Theorem (G '06)

If Γ is a locally finite connected graph then $|\Gamma^3|$ has a Hamilton circle.

in G: "Infinite hamilton cycles in squares of locally finite graphs", Preprint 2007;

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- solve some of the open problems.