From mafia expansion to analytic functions in percolation theory

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Joint work with John Haslegrave,
and with Christoforos Panagiotis

A model for Mafia growth

A “social” network evolves in
(continuous or discrete)
time according to the following rules

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Is $M(\lambda)$ finite or infinite?
It is finite almost surely

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finite in the synchronous case,
we don't know in the asynchronous case

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How does the expected size depend on λ?

The expected size of $M(\lambda)$

Let $\chi(\lambda) := \mathbb{E}(|M(\lambda)|)$

Theorem (G & Haslegrave '18+)

$$e^{c\lambda} \leq \chi(\lambda) \leq e^{e^{C\lambda}}$$

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Is $\chi(\lambda)$ continuous in λ?

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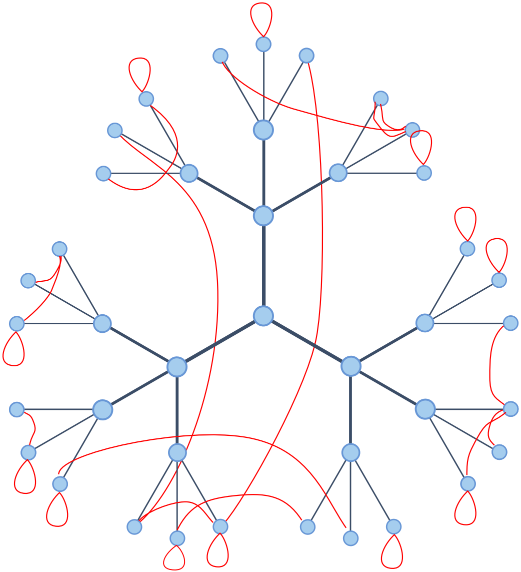
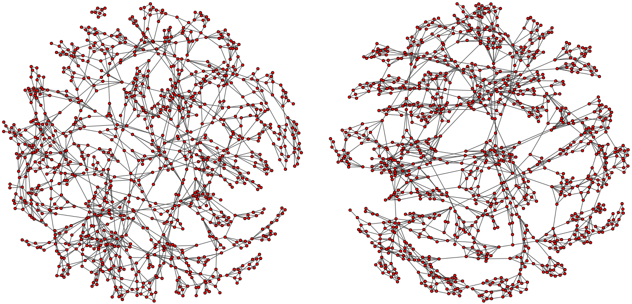
Percolation ...

Random Graphs from trees



Random Graphs from trees

Simulations by C. Moniz.



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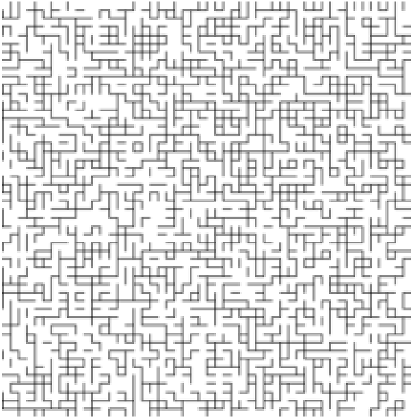
Theorem (Newman & Schulman, Aizenman & Newman '86)

In long range percolation on $\mathbb{Z}$, percolation occurs for large enough λ iff $s \leq 2$.

Percolation model



Bernoulli bond percolation on an infinite graph, i.e.

Each edge
-present with probability p,
and
-absent with probability $1 - p$
independently of other edges.

Percolation threshold:

$$p_c := \sup\{p \mid \mathbb{P}_p(\text{ component of } o \text{ is infinite }) = 0\}$$

Historical remarks on percolation theory

Classical era:
Introduced by physicists Broadbent & Hammersley '57

$p_c(\text{square grid}) = 1/2$ (Harris '59 + Kesten '80)

Many results and questions on phase transitions, continuity, smoothness etc. in the '80s:
Aizenman, Barsky, Chayes, Grimmett, Hara, Kesten, Marstrand, Newman, Schulman, Slade, Zhang ... (apologies to many!)

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Thought of as part of statistical mechanics

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... for example, percolation can characterise amenability:

Theorem ($\Leftarrow$ Aizenman, Kesten & Newman ’87, $\Rightarrow$ Pak & Smirnova-Nagnibeda ’00)

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A finitely generated group is non-amenable iff it has a Cayley graph with $p_c < p_u$.

See the textbooks [*Lyons & Peres '15*], [*Pete '18+*] for more.

Back to classics: analyticity below p_c

$\chi(p) := \mathbb{E}_p(|C(o)|)$,

i.e. the expected size of the component of the origin o.

Theorem (Kesten '82)

$\chi(p)$ is an analytic function of p for $p \in [0, p_c)$ when G is a lattice in $\mathbb{R}^d$.

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Proved by extending p and $\chi(p)$ to the complex numbers, and using classical complex analysis (Weierstrass).

Some complex analysis basics

Theorem (Weierstrass): Let $f = \sum f_n$ be a series of analytic functions which converges uniformly on each compact subset of a domain $\Omega \subset \mathbb{C}$. Then f is analytic on Ω.

Weierstrass M-test: Let (f_n) be a sequence of functions such that there is a sequence of 'upper bounds' M_n satisfying

$$|f_n(z)| \leq M_n, \forall x \in \Omega \qquad \text{and} \qquad \sum M_n < \infty.$$

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Theorem (Aizenman & Barsky '87)

In every vertex-transitive percolation model,

$$\mathbb{P}_p(|C| > n) \leq c_p^{-n},$$

for every $p < p_c$ and some $c_p > 1$.

Conjectures on the percolation probability

$\theta(p) := \mathbb{P}_p(|C| = \infty)$,

i.e. the percolation probability.

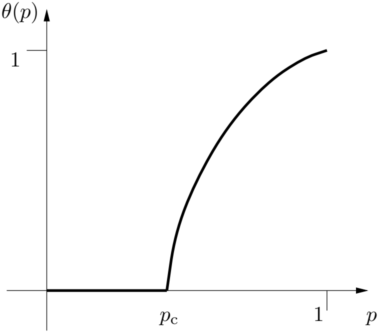

Fig. 1.1. It is generally believed that the percolation probability $\theta(p)$ behaves roughly as indicated here. It is known, for example, that θ is infinitely differentiable except at the critical point p_c. The possibility of a jump discontinuity at p_c has not been ruled out when $d \geq 3$ but d is not too large.

$\theta(p)$ analytic?

Open problem:

Is $\theta(p)$ analytic for $p > p_c$?

Appearing in the textbooks Kesten '82, Grimmett '96, Grimmett '99.

Our results (G & Panagiotis '18+)

- θ etc. analytic for $p > p_c$ on regular trees.
  –trivial for binary tree, but what about higher degrees?

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  – Braga et.al. '04 prove analyticity near $p = 1$ for $\mathbb{Z}^d$
- $p_{\mathbb{C}} \leq 1/2$ on certain families of triangulations.
  – progress on questions of Benjamini & Schramm '96, and Benjamini '16.

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'...this in not just an academic matter. For instance, there are examples of disordered systems in statistical mechanics that develop a Griffiths singularity, i.e., systems that have a phase transition point even though their free energy is a C^∞ function.'
–Braga, Proccaci & Sanchis '02

Partitions of n

Theorem (Hardy & Ramanujan 1918)

The number of partitions of the integer n is of order

$$exp(\sqrt{n}).$$

Elementary proof: [*P. Erdös, Annals of Mathematics '42*]

Finitely presented Cayley graphs

Theorem: $p_{\mathbb{C}} < 1$ for every finitely presented Cayley graph.

Similar arguments, but we had to generalise *separating curves* to all graphs.

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These slides are on-line

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