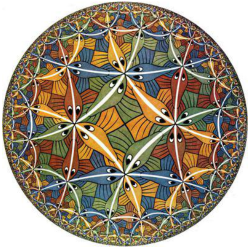
The planar Cayley graphs are effectively enumerable

Agelos Georgakopoulos

THE UNIVERSITY OF WARWICK



Neuchatel, 20/10/15

Groups need to act!

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Let them act on the plane

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and be finitely generated

Planar discontinuous groups

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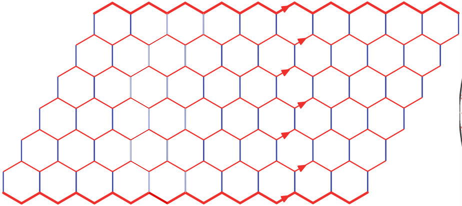
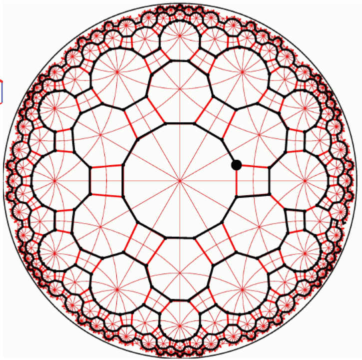

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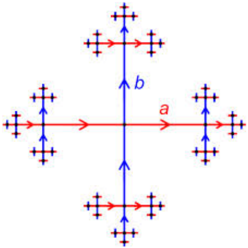

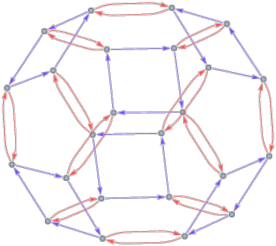

Known facts

Planar discontinuous groups

- admit planar Cayley graphs
- are virtually surface groups
- admit one-relator presentations
- are effectively enumerable

see [*Surfaces and Planar Discontinuous Groups, Zieschang, Vogt & Coldewey; Lecture Notes in Mathematics*]

or [*Lyndon & Schupp*].

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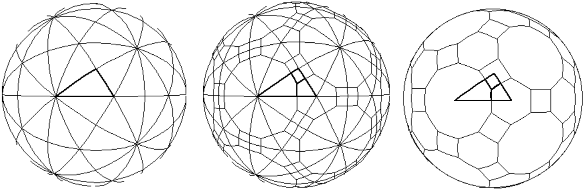

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Are Planar discontinuous groups exactly those having a planar Cayley graph? **NO!**

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Are Planar discontinuous groups exactly those having a planar Cayley graph? **NO!**

Definition: a group is planar, if it has a planar Cayley graph.

Charactisation of the finite planar groups

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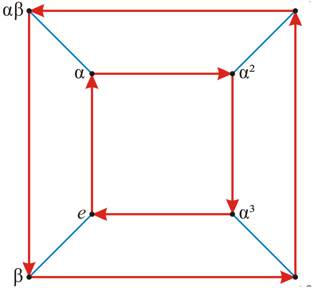
Theorem (Maschke 1886)

Every finite planar group is a group of isometries of S^2.

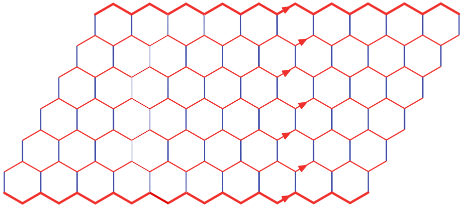


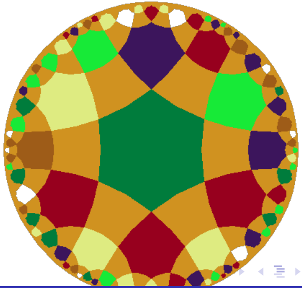
The 1-ended planar groups

Theorem ((classic) Macbeath, Wilkie, ...)

Every 1-ended planar Cayley graph corresponds to a group of isometries of $\mathbb{R}^2$ or $\mathbb{H}^2$.

see [*Surfaces and Planar Discontinuous Groups, Zieschang, Vogt & Coldewey; Lecture Notes in Mathematics*]





The Cayley complex

Theorem (G '12, Known?)

A group has a flat Cayley complex if and only if it has a accumulation-free Cayley graph.

(In which case it is a planar discontinuous group.)

Facial presentations

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A facial presentation is a triple $(\mathcal{P} =< \mathcal{S} \mid \mathcal{R} >, \sigma, \tau)$, where

- σ is a spin, i.e. a cyclic ordering on $\mathcal{S}$, and
- $\tau : \mathcal{S} \to \{T, F\}$ decides which generators are spin-preserving or spin-reversing, so that
- every relator is a facial word.

Facial presentations

Theorem (G '12)

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based on...

Theorem (Whitney '32)

Let G be a 3-connected plane graph. Then every automorphism of G extends to a homeomorphism of the sphere.

... in other words, every automorphism of G preserves facial paths.

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planar Cayley graphs with accumulation points

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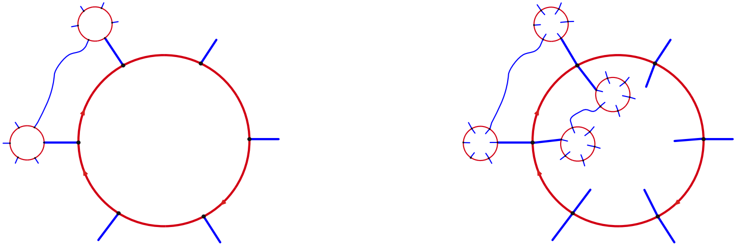

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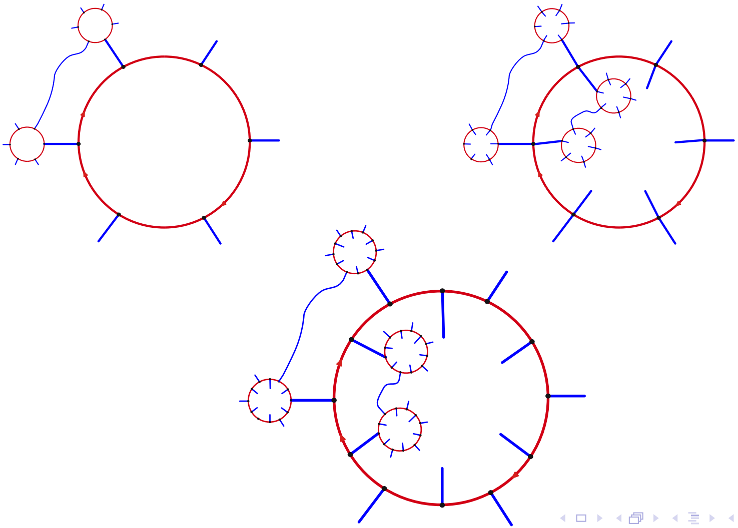

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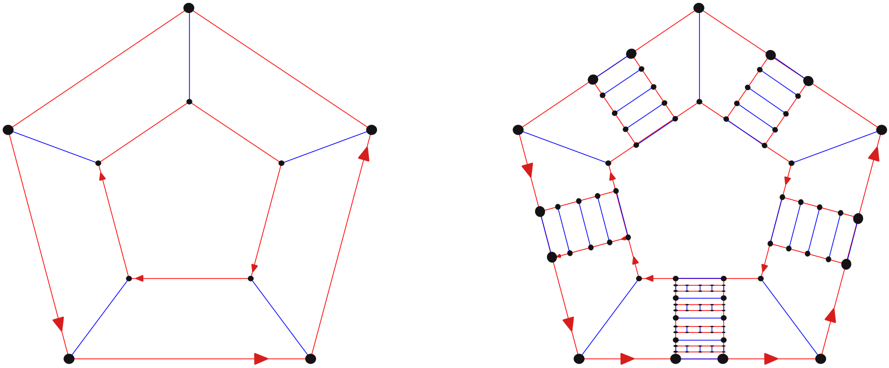

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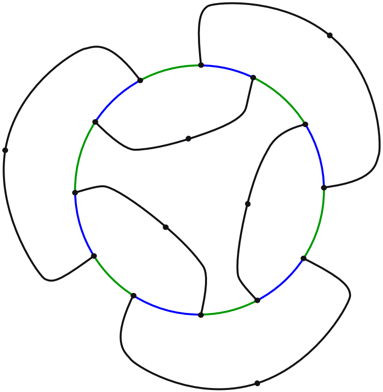

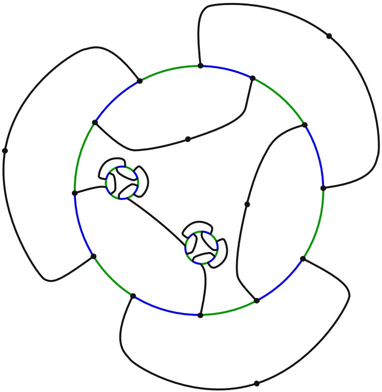

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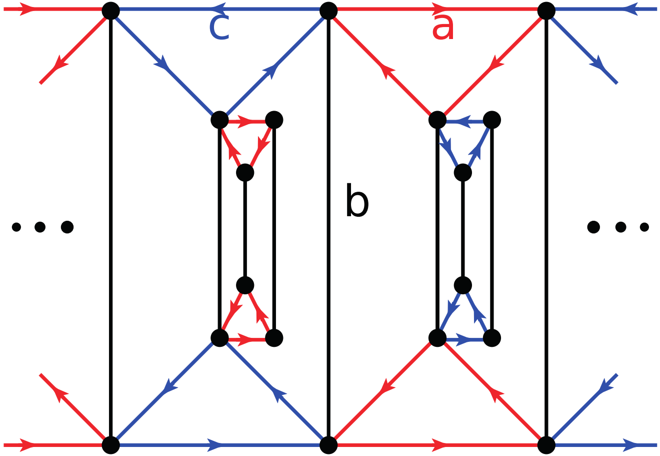

What we didn't know

Open Problems:

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Problem (Mohar)

How can you split a planar Cayley graph with > 1 ends into simpler Cayley graphs?

Problem (Droms et. al.)

Is there an effective enumeration of the planar locally finite Cayley graphs?

Problem (Bonnington & Watkins (unpublished))

Does every planar 3-connected locally finite transitive graph have at least one face bounded by a cycle.

... and what about all the classical theory?

Dunwoody's theorem

Theorem (Dunwoody '09)

If Γ is a group and G is a connected locally finite planar graph on which Γ acts freely so that Γ/G is finite, then Γ or an index two subgroup of Γ is the fundamental group of a graph of groups in which each vertex group is either a planar discontinuous group or a free product of finitely many cyclic groups and all edge groups are finite cyclic groups (possibly trivial).

Classification of the cubic planar Cayley graphs

Theorem (G '10, to appear in Memoirs AMS)

Let G be a planar cubic Cayley graph. Then G is colour-isomorphic to precisely one element of **the list***.*

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Let G be a planar cubic Cayley graph. Then G is colour-isomorphic to precisely one element of **the list**.
Conversely, for every element of the list and any choice of parameters, the corresponding Cayley graph is planar.

Presentations of planar Cayley graphs with accumulation points?

Recall that every accumulation-free Cayley graph has a facial presentation.

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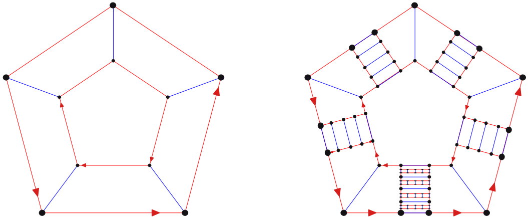

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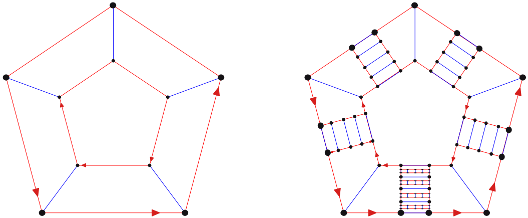

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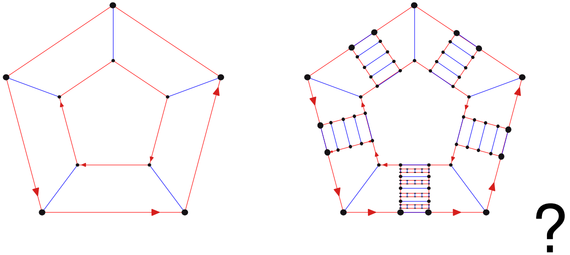

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How do we generalise?

Planar presentations

A presentation $\mathcal{P} =< \mathcal{S} \mid \mathcal{R} >$ is planar, if it can be endowed with spin data σ, τ so that

- no two relator words cross
- every relator contains an even number of spin-reversing letters.

σ is a spin, i.e. a cyclic ordering on $\mathcal{S}$
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Cheat: this is a simplified definition, corresponding to the 3-connected case;
The general (2-connected) case is much harder to state and prove.

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The proof of forward direction involves ramifications of Dunwoody cuts. The proof of the backward direction is elementary, and mainly graph-theoretic, but hard.

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Two steps:

—Step 1: if C comes from a relator W

—Step 2: for general C, write $C = \sum W_i$, and apply Step 1 to each W_i.

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Let's still try:

$$I_C := I_1 \triangle I_2 \triangle \ldots I_k$$
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Suppose it works;

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Suppose it works; then anything works!

The Theorem

Theorem (G & Hamann, '14)

A Cayley graph G is planar iff it admits a planar presentation.

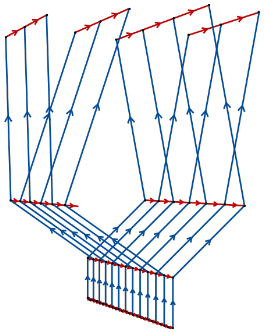
Corollary

The planar groups are effectively enumerable.

(Answering Droms et. al.)

Outlook

Generalise to include



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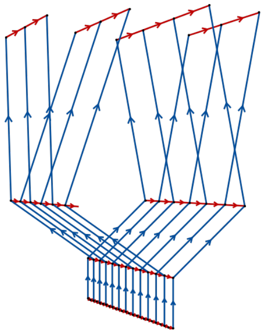


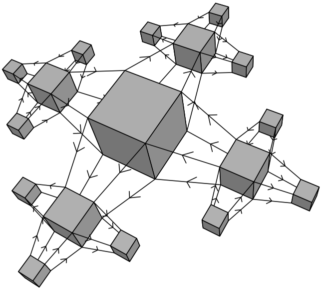
Does anybody know if the groups having a Cayley complex embeddable in $\mathbb{R}^3$ have been characterised?

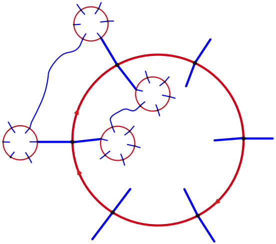
Outlook

Theorem (Stallings '71)

Every group with >1 ends can be written as an HNN-extension or an amalgamation product over a finite subgroup.

Can we generalise this to graphs?





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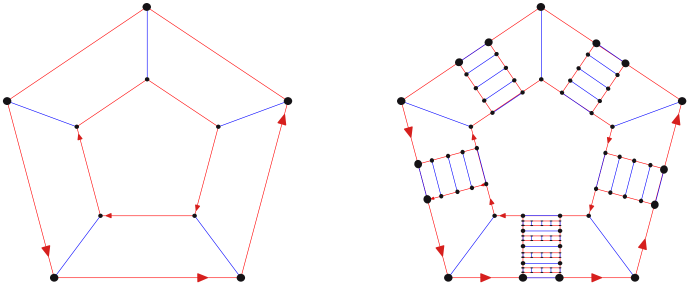

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