On the expected size of a patriarchal mafia with Poisson distributed collaborations

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THE UNIVERSITY OF WARWICK

Rome, 23/2/17

Partly joint work with J. Haslegrave

Disclaimer: *The topic of this talk has nothing to do with the location*

How Mafia's grow

A network evolves in (continuous or discrete) time with the following rules:

- When a (Poisson) clock ticks, nodes split into two;
- When a node x splits into two nodes x', x'', each of its existing edges gets inherited by x' or x'' independently with probability 1/2;
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If it is finite, is its expected size finite or infinite?
If finite, how does it depend on k?

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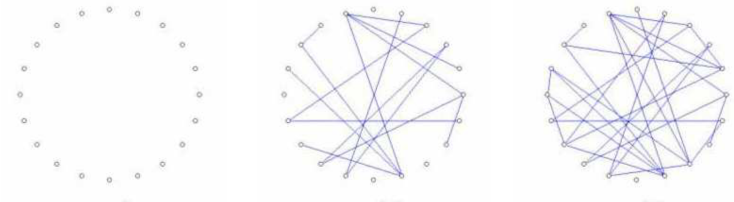
- each pair joined with an edge, independently, with same probability $p = p(n)$.



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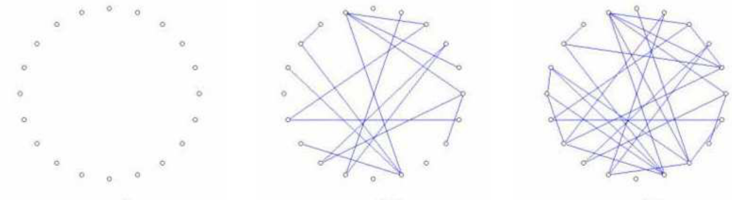

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Real-world networks?

- Preferential attachment networks
- Geometric random graphs

Geometric Random Graphs Literature

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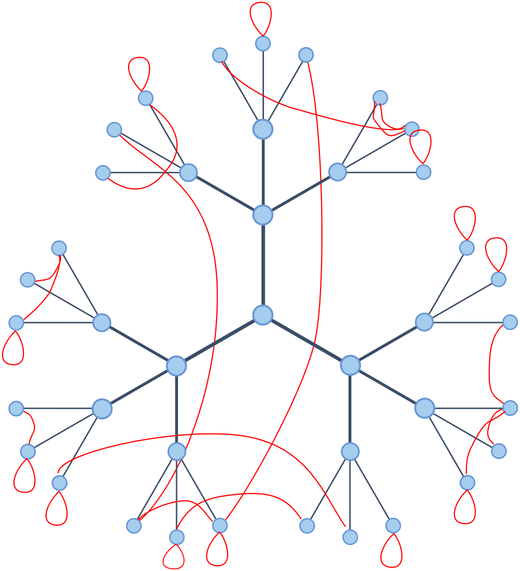
Percolation theory ...

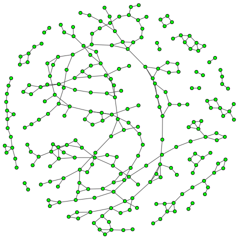
Random Graphs from trees



$R_3^1(T)$

Random Graphs from trees

Simulations by C. Moniz.



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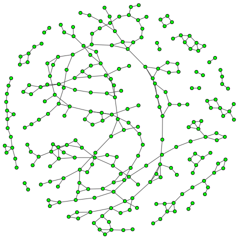
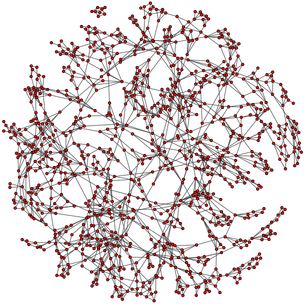
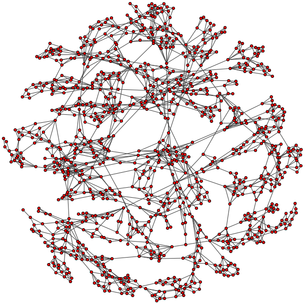
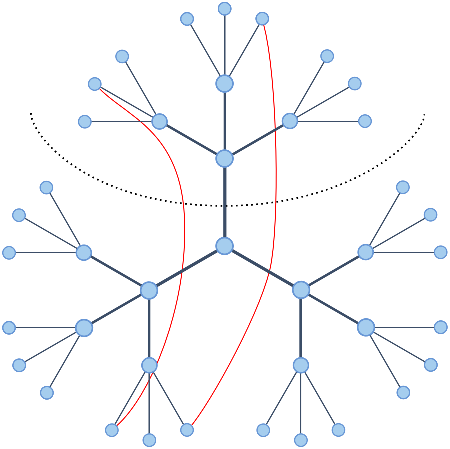






A nice property



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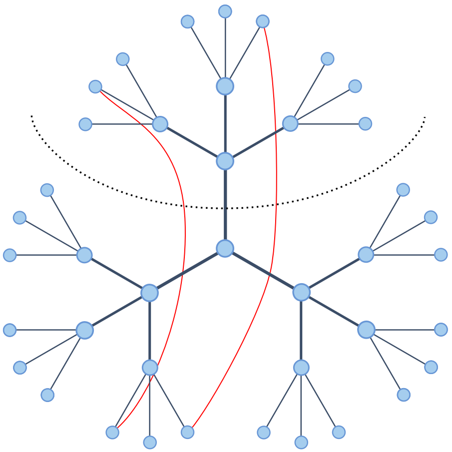

Proposition

$\mathbb{E}(\sharp$ edges xy in R_n with x in X and y in $Y)$ converges.

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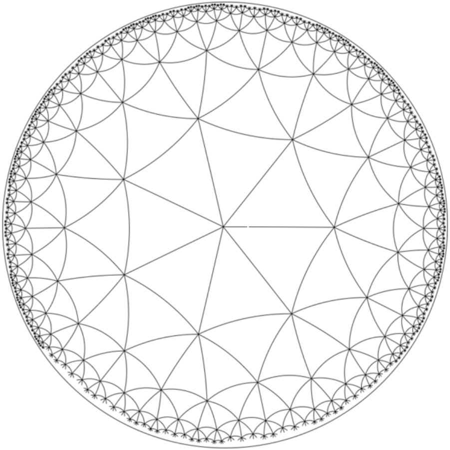

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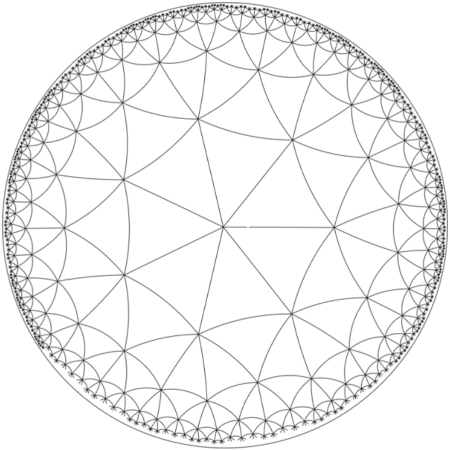

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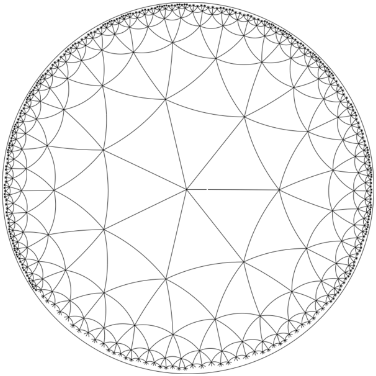

Proposition

For every two measurable subsets X, Y of the Poisson (or Martin) boundary ∂G,

$$\mathbb{E}(\sharp \textit{ edges } xy \textit{ in } R_n \textit{ with } x \textit{ 'close to' } X \textit{ and } y \textit{ 'close to' } Y)$$

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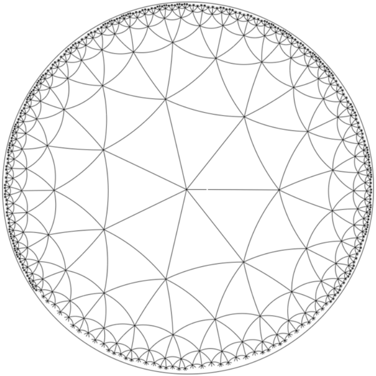

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We use the limit to define a measure on $\partial G \times \partial G$ via

$$C(X, Y) := lim \mathbb{E}(\sharp \text{ edges } ...)$$

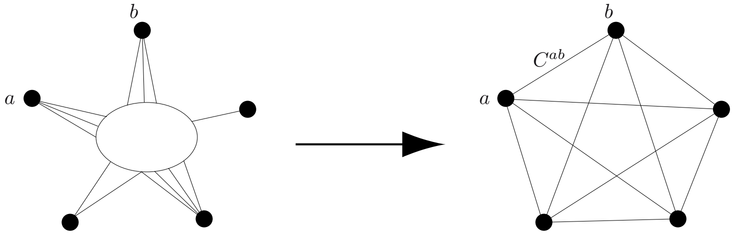
Energy and Douglas' formula

The classical Douglas formula [Douglas '31]

$$E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(\zeta, \eta) d\eta d\zeta$$

calculates the (Dirichlet) energy of a harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

Energy in finite electrical networks



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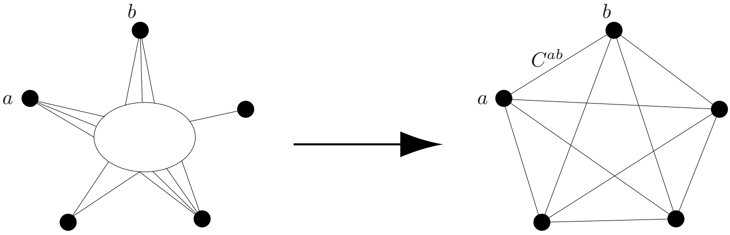

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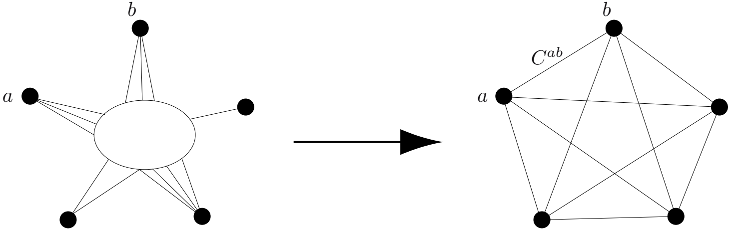

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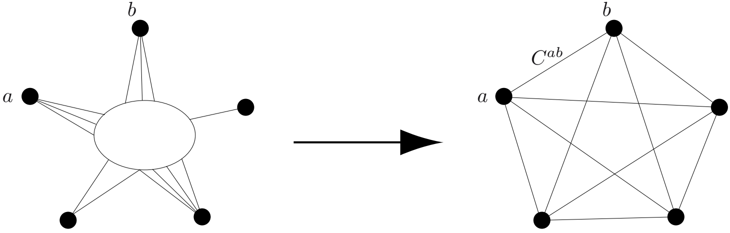

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How can we generalise this to an arbitrary domain?
To an infinite graph?

Effective conductance

We call C the *effective conductance measure*, because

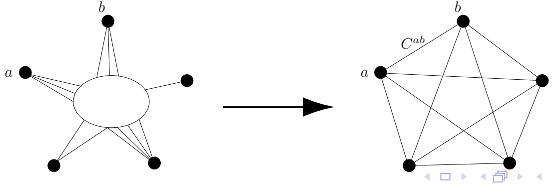
Theorem (G & V. Kaimanovich '12-'17+)

For every locally finite network G, and every harmonic function h, we have

$$E(h) = \int_{\partial G \times \partial G} \left(\widehat{h}(\eta) - \widehat{h}(\zeta)\right)^2 dC(\eta, \zeta).$$

History: Douglas '31, Naim '57, Doob '62, Silverstein '74

Finite version: $E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 \, C_{ab}$



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Theorem (G & Kaimanovich '17+)

For every transient, locally finite graph G,

$$C(X, Y) = \nu(1_{XY} W^*).$$

Long range percolation

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How large is $R_\infty^\lambda(T)$?

The expected size of the TWRG

Let C_o^λ denote the component of a uniformly random vertex of $R_n^\lambda(T)$ (or $R_\infty^\lambda(T)$).

Theorem (G & Haslegrave, state of the art 2/17)

$$e^\lambda \leq \mathbb{E}(|C_o^\lambda|) \leq e^{e^{b\lambda}}.$$

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Conjecture:

$$\mathbb{E}(|C_o^\lambda|) \sim \lambda^\lambda$$

(backed by simulations)

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Thank you!

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