Group Walk Random Graphs

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Random Graphs flashback

1269 papers on MathSciNet with "random graph" in their title

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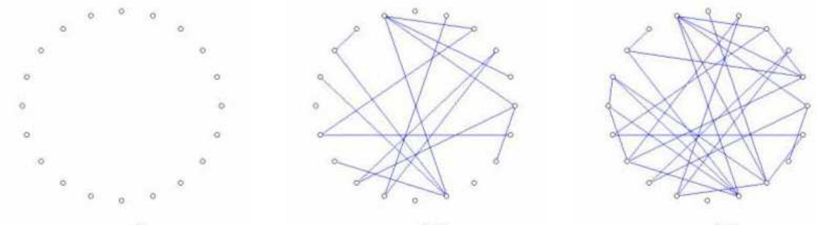
... most of which on the Erdős-Renyi model $G(n,p)$:

- n vertices
- each pair joined with an edge, independently, with same probability $p = p(n)$.



$p = 0$ $p = 0.1$ $p = 0.2$

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1. [Gilbert, E. N. Random graphs. Ann. Math. Statist. 30 1959]
   => determines the probability that the graph is connected.

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⋮

10. [Palásti, I. On the connectedness of random graphs. Studies in Math. Stat.: Theory & Applications. 1968]
   => gives a short summary of some previously published results concerning the connectedness of random graphs.

⋮

Random Graphs flashback

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⋮

100. [Bollobás, B. Long paths in sparse random graphs. Combinatorica. 1982]

=> shows that if $p = c/n$, then almost every graph in $G(n, p)$ contains a path of length at least $(1 - a(c))n$, where $a(c)$ is an exponentially decreasing function of c.

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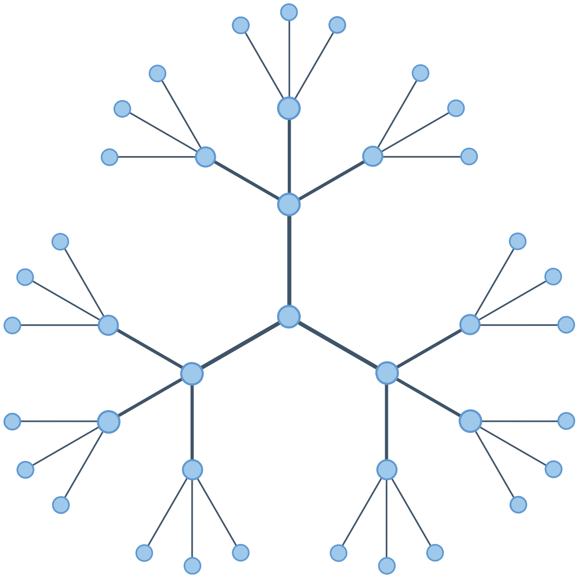
1000. [Doku-Amponsah, K.; Mörters, P. Large deviation principles for empirical measures of colored random graphs. Ann. Appl. Probab. 2010]

=> derives large deviation principles for the empirical neighbourhood measure of colored random graphs, defined as the number of vertices of a given colour with a given number of adjacent vertices of each colour. ...

Random Graphs from trees



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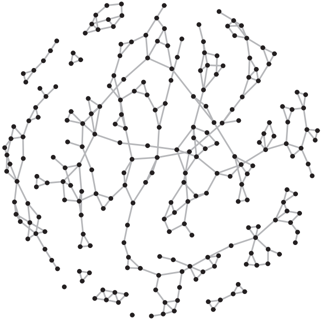
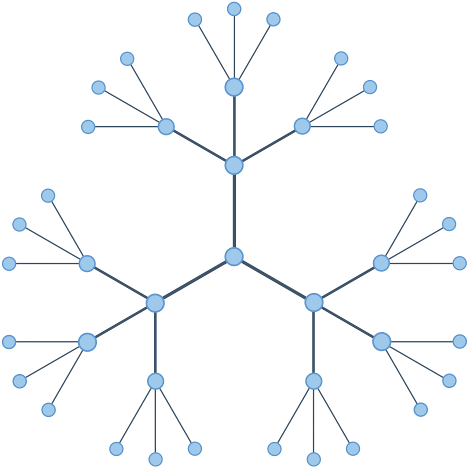


Simulation on the binary tree by A. Janse van Rensburg.

A nice property



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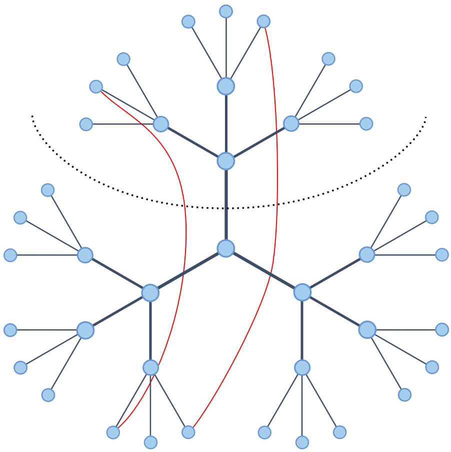

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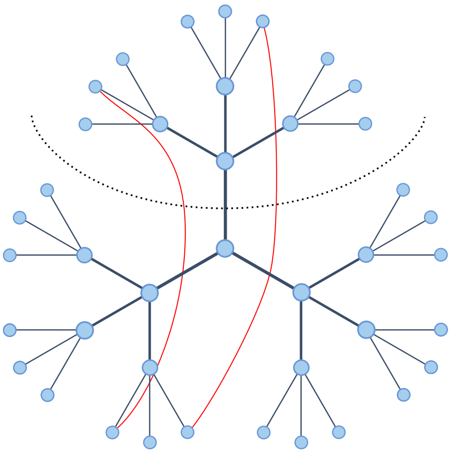

Proposition

$\mathbb{E}(\sharp$ *edges xy in* $\mathcal{G}_n(T)$ *with x in X and y in Y*$)$

converges.

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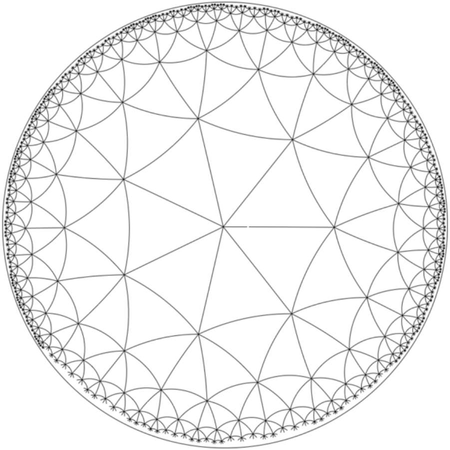

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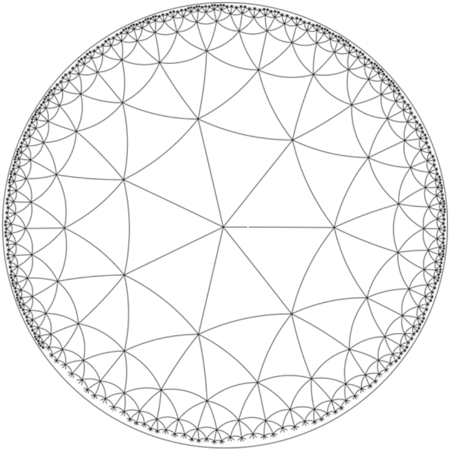

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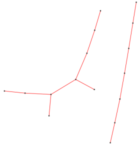
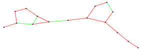
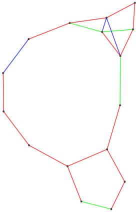
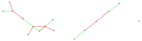
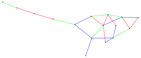
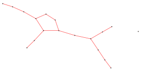
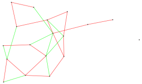
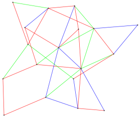
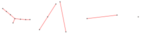
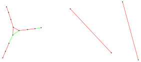
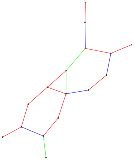
?

	One set	Two sets	Three sets
Cycled Binary Tree			
2Grid			
3Grid			
Lamplighter			

Simulations by C. Midgley.

Problems

Problem 1: The (expected) number of connected components (or isolated vertices) is asymptotically proportional to $|B_n|$.

Problem 2: The expected diameter of the largest component is asymptotically $c \log |B_n|$.

Backed by simulations by C. Midgley.

What's the point?

Metaproblem 1: Which properties of the random graphs are determined by the group of the host graph H and do not depend on the choice of a generating set?

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Metaproblem 2: Which group-theoretic properties of the host group are reflected in graph-theoretic properties of the random graphs?

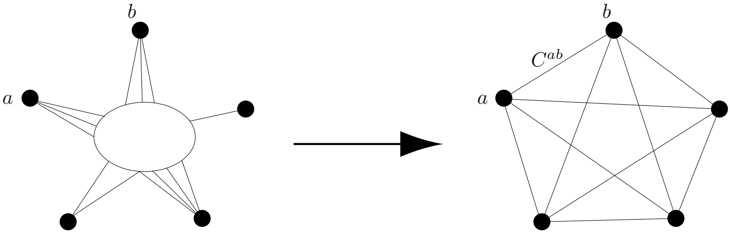
Energy and Douglas' formula

The classical Douglas formula

$$E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(\zeta, \eta) d\eta d\zeta$$

calculates the (Dirichlet) energy of a harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

Energy in finite electrical networks



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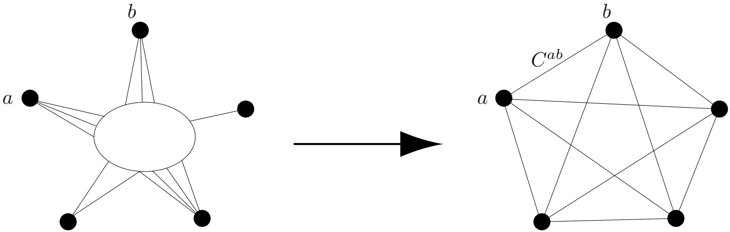

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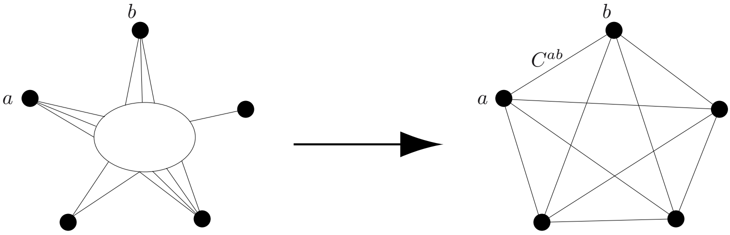

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How can we generalise this to an arbitrary domain?

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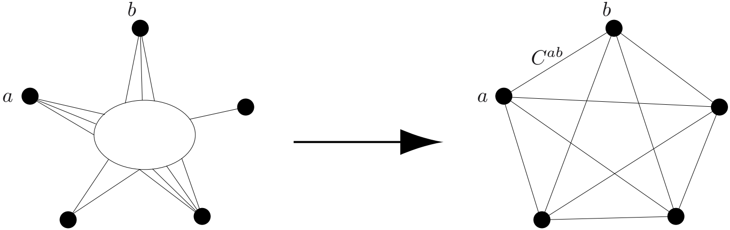

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How can we generalise this to an arbitrary domain?
To an infinite graph?

The Poisson integral representation formula

The classical Poisson formula

$$h(z) = \int_0^1 \hat{h}(\theta) P(z,\theta) d\theta$$

where $P(z,\theta) := \frac{1-|z|^2}{|e^{2\pi i\theta}-z|^2}$,
recovers every continuous harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

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- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;
- conversely, for every $\hat{h} \in L^\infty(\mathcal{P}_G)$ the function $z \mapsto \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$ is bounded and harmonic.

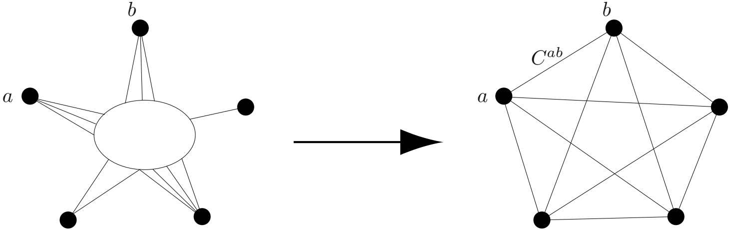
i.e. there is Poisson-like formula establishing an isometry between the Banach spaces $H^\infty(G)$ and $L^\infty(\mathcal{P}_G)$.

The Poisson-Furstenberg boundary

Selected work on the Poisson boundary

- Introduced by Furstenberg to study semi-simple Lie groups [*Annals of Math. '63*]
- Kaimanovich & Vershik give a general criterion using the entropy of random walk [*Annals of Probability '83*]
- Kaimanovich identifies the Poisson boundary of hyperbolic groups, and gives general criteria [*Annals of Math. '00*]

Energy in finite electrical networks



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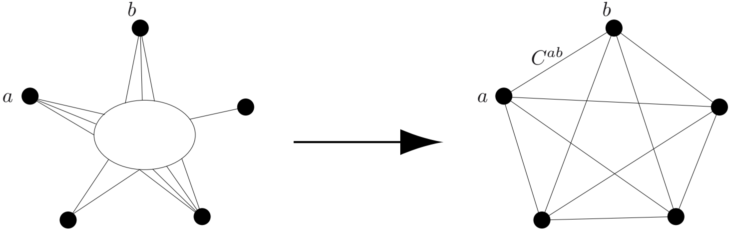

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[*Doob '62*] generalises this to Green spaces (or Riemannian manifolds) using their *Martin boundary*.

The energy of harmonic functions

Theorem (G & Kaimanovich '14+)

For every locally finite network G, there is a measure C on $\mathcal{P}^2(G)$ such that for every harmonic function u the energy $E(u)$ equals

$$\int_{\mathcal{P}^2} \left(\widehat{u}(\eta) - \widehat{u}(\zeta)\right)^2 dC(\eta, \zeta).$$

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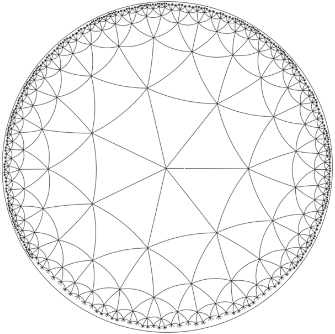

$$C(X, Y) := \lim_n \mathbb{E}(\sharp \text{ edges } xy \text{ in } \mathcal{G}_n(H)$$
$$\text{with } x \text{ 'close to' } X,$$
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where the **Naim Kernel** Θ is defined as

$$\Theta(\zeta, \eta) := \frac{1}{G(o,o)} \lim_{z_n \to \zeta, y_n \to \eta} \frac{F(z_n, y_n)}{F(z_n, o) F(o, y_n)}$$

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Remark:

$$\frac{1}{\Theta(z, y)} = G(o, o) \Pr_z(o < y \mid y),$$

where $\Pr_z(o < y|y)$ is the conditional probability to visit o before y subject to visiting y.

Convergence of the Naim Kernel

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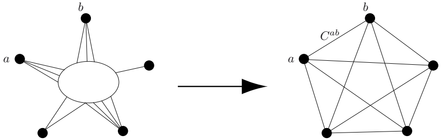
Problem: Let $(z_i)_{i\in\mathbb{N}}$ and $(w_i)_{i\in\mathbb{N}}$ be independent simple random walks from o. Then $\lim_{n,m\to\infty}\Theta(z_n,w_m)$ exists almost surely.

Θ in finite electrical networks



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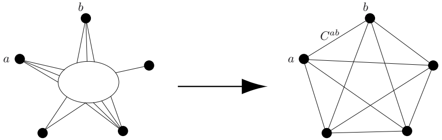

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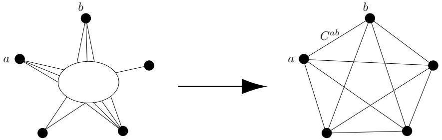

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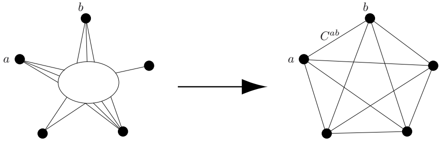

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Proposition

For every measurable $X, Y \subseteq \mathcal{P}(G)$

$$C_n(X,Y) = \mathbb{E}(\Theta^n(x_n, y_n) 1_{XY}).$$

Therefore, $C(X,Y) = \lim_n \mathbb{E}(\Theta^n(x_n, y_n) 1_{XY})$.

Random Interlacements and C

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- applied to study the vacanct set on the discrete 3D-torus
- governed by a certain σ-finite measure ν

Random Interlacements and C

Random Interlacements $\mathcal{I}$ [Sznitman]:

- A Poisson point process whose ‘points’ are 2-way infinite trajectories
- applied to study the vacanct set on the discrete 3D-torus
- governed by a certain σ-finite measure ν

Claim: $C(X, Y) = \nu(1_{XY} W^*)$.

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The effective conductance measure C,
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Can we use them to study groups?