On well-quasi-ordering rayless graphs

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Abstract

In the aftermath of the Robertson–Seymour Graph Minor Theorem, Thomas conjectured that the countable graphs are well-quasi-ordered under the minor relation. We prove that this conjecture, even when restricted to rayless graphs, would imply that the finite graphs are better-quasi-ordered, another well-known open problem.

Motivated by this implication we then focus on Thomas' conjecture for rayless graphs, and prove several equivalent reformulations, one of which being that the rayless countable graphs of rank α can be decomposed into exactly $\aleph_0$ minor-twin classes for every ordinal $\alpha < \omega_1$.

By restricting the latter statement to trees, and combining it with Nash-Williams' theorem that the infinite trees are well-quasi-ordered, we deduce as a side result that a minor-closed family of $\mathbb{N}$-labelled rayless forests is Borel if and only if it does not contain all rayless forests.

As another side-result, we prove Seymour's self-minor conjecture for rayless graphs of any cardinality.

Keywords: graph minor, well-quasi-ordered, better-quasi-ordered, rayless, Borel set, self-minor conjecture.

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1 Introduction

The celebrated Robertson–Seymour Graph Minor Theorem [14]–[15] states that the finite graphs are well-quasi-ordered under the minor relation $<$. This paper focuses on two well-known problems that it left open:

Conjecture 1.1 (Folklore [10]). *The finite graphs are better-quasi-ordered (BQO) under the minor relation.*

Conjecture 1.2 (Thomas' conjecture [18]). *The countable graphs are well-quasi-ordered (WQO) under the minor relation.*

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See [10] for a survey of the notions WQO and BQO and some motivation. According to Pequignot [10], the poset of finite graphs endowed with the minor relation is

"the only naturally occurring WQO which is not yet known to be BQO".

Our first result provides a connection between the above conjectures:

Theorem 1.1. *If the countable rayless[1] graphs are WQO, then the finite graphs are BQO.*

We will also prove a partial converse ((10)), thereby establishing that the finite graphs are BQO if and only if a certain subfamily of the countable rayless graphs is WQO. In other words, we have reduced the better-quasi-ordering of the finite graphs to a 'simple' statement about infinite graphs.

Commenting on Thomas' Conjecture 1.2 and similar questions, Robertson, Seymour, & Thomas [13] wrote:

> "There is not much chance of proving these conjectures because they imply that the set of all finite graphs is 'second-level better-quasi-ordered' by minor containment, which in itself seems to be a hopelessly difficult problem".

While this still seems up-to-date, the results of this paper provide new tools for attacking Thomas' conjecture, and point out interesting special cases. Perhaps there is now a chance of disproving it (see Section 11).

But let me first try to explain the above remark. What is meant by 'second-level better-quasi-ordered' here is probably the following. Given two sets of graphs $\mathcal{G}, \mathcal{G}'$, we write $\mathcal{G} \preceq \mathcal{G}'$ if for every $G \in \mathcal{G}$ there is $H \in \mathcal{G}'$ such that $G < H$.

Problem 1.3. *Are the sets of finite graphs well-quasi-ordered under $\preceq$?*

A positive answer would imply that the minor-closed families of finite graphs are well-quasi-ordered by the inclusion relation, and this provides strong motivation for the problem.

The connection between Problem 1.3 and Conjecture 1.1 is that one can take the definition of $\preceq$ further, and apply it to sets of sets of graphs, and iterate transfinitely, in order to define better-quasi-ordering; see Section 2.4 for more.

In particular, our Theorem 1.1 implies that if Thomas' Conjecture 1.2 has a positive answer, then so does Problem 1.3, and in fact the restriction of Thomas' conjecture to rayless graphs has the same implication.

Motivated by Theorem 1.1, we undertake a study of Thomas' conjecture for rayless graphs, noting that some long-standing open problems for infinite graphs have been settled in the rayless case [1, 11]. As a warm-up, we will prove Seymour's (unpublished) self-minor conjecture for rayless graphs of any cardinality, which to the best of my knowledge was open even for countable graphs:

Corollary 1.2. *For every infinite rayless graph G, there is a minor model of G in itself which is not the identity.*

[1] A graph is *rayless*, if it does not contain an (1-way) infinite path.

(Oporowski [9] has found uncountable counterexamples to Seymour's self-minor conjecture, which of course contain rays.)

A well-known equivalent way to define what it means for a quasi-order $(Q, \leqslant)$ to be well-quasi-ordered is to say that it has no infinite antichain and no infinite descending chain. None of these conditions implies the other in general, but we will show that in our setup they are in fact equivalent. Let $\mathcal{R}$ denote the class of countable rayless graphs. Our main result can be summarized as follows:

Theorem 1.3. *The following are equivalent:*

(a) $\mathcal{R}$ *is well-quasi-ordered;*

(b) $\mathcal{R}$ *has no infinite descending chain;*

(c) $\mathcal{R}$ *has no infinite antichain;*

(d) $\mathcal{R}$ *consists of exactly* $\aleph_1$ *minor-twin equivalence classes.*

The most interesting aspect of Theorem 1.3 is the equivalence of the cardinality condition (d) to the well-quasi-ordering of $\mathcal{R}$, which I will now explain. We say that two graphs G, H are *minor-twins*, if both $G < H$ and $H < G$ hold. Each rayless graph can be assigned an ordinal number, called its *rank*, by recursively decomposing G into graphs of smaller ranks (Section 2.3) similarly to the definition of rank for Borel sets. It is known that every ordinal α smaller that the least uncountable ordinal ω_1 is the rank of some countable graph G. Thus an alternative way to formulate (d) is to say that for every $\alpha < \omega_1$, the rayless countable graphs of rank α form exactly $\aleph_0$ minor-twin classes. In fact, we will prove a refinement of Theorem 1.3 (Theorem 9.1) whereby $\mathcal{R}$ is replaced by the subclass of graphs of (up to) a given rank. This is achieved via an intricate transfinite induction in which we have to show all of these conditions to be equivalent before being able to proceed to the next rank. In fact, we will have to introduce additional equivalent conditions for the induction to work, which involve graphs that have a finite set of their vertices *marked*, endowed with a *marked minor* relation that maps each marked vertex to a branch set containing at least one marked vertex (Section 2.2).

I expect that Theorem 1.3 remains true when replacing $\mathcal{R}$ by a variety of subclasses, e.g. planar graphs. For the case of rayless trees, we will prove the implication (a) $\to$ (d). Combining this with Nash-Williams' [8] theorem that the trees are well-quasi-ordered[2], we deduce as above that there are only countably many minor-twin classes of countable rayless trees of rank α for every $\alpha < \omega_1$. Combining this with ongoing work with J. Grebík [4] connecting minor-closed families and Borel subsets of the space $\mathcal{G}$ of $\mathbb{N}$-labelled graphs[3], we will deduce the following.

Theorem 1.4. *Let* $\mathcal{T} \subset \mathcal{G}$ *be a minor-closed family of* $\mathbb{N}$*-labelled rayless forests. Then* $\mathcal{T}$ *is Borel if and only if it is proper, i.e. it does not contain all rayless forests.*

[2]In fact, we will need a strengthening of this, proved by Thomas, saying that the graphs of tree-width k are well-quasi-ordered for every $k \in \mathbb{N}$.

[3]$\mathcal{G}$ denotes the space of graphs G with $V(G) = \mathbb{N}$, encoded as functions from $\mathbb{N}^2$ to $\{0, 1\}$ representing the edges, endowed with the Tychonoff topology. The vertex labelling is ignored when considering minors; it is only used to define the topology on $\mathcal{G}$.

This paper is structured as follows. After some preliminaries, we prove Theorem 1.1 in Section 3. Most of the rest of the paper is devoted to Theorem 1.3, and we prepare it with a warm-up: Section 4 focuses on graphs of rank 1, introducing some fundamental ideas of the paper, and concluding with Corollary 1.2. Section 5 is devoted to a key lemma (Lemma 5.1) implying that, under mild conditions, the minor-twin class of a rayless graph G of rank α is determined by the marked-minors of G of ranks $< \alpha$. Sections 6 and 7 constitute an Intermezzo devoted to the proof of Theorem 1.4, making use of the aforementioned lemma (this constitutes an early example of a statement about unmarked graphs for the proof of which marked graphs are necessary). After this we return to the proof of Theorem 1.3, introducing some un-marking techniques in Section 8, followed by the main technical part in Section 9, and concluding with the wrapping-up Section 10. We finish with some open problems in Section 11.

2 Preliminaries

We will be following the terminology of Diestel [3] for graph-theoretic concepts.

2.1 (Well)-Quasi-Orders

A quasi-order $(Q, \leqslant)$ consists of a set Q and a binary relation $\leqslant$ on Q which is reflexive and transitive (but not necessarily antisymmetric). A quasi-order $(Q, \leqslant)$ is said to be *well-quasi-ordered*, if for every sequence $(G_n)_{n\in\mathbb{N}}$ of its elements there are $i < j$ such that $G_i \leqslant G_j$. If such i, j exist then we say that $(G_n)_{n\in\mathbb{N}}$ is *good*, otherwise it is *bad*.

A well-known consequence of Ramsey's theorem is

Proposition 2.1 ([3, Proposition 12.1.1])**.** *$(Q, \leqslant)$ is WQO if and only if it has no infinite antichain and no infinite sequence $(G_n)_{n\in\mathbb{N}}$ such that $G_{n+1} \leqslant G_n$ and $G_n \nleqslant G_{n+1}$ for every n.*

Such a sequence $(G_n)_{n\in\mathbb{N}}$ is called a *descending chain*. The following is also well-known:

Observation 2.2 ([3, Corollary 12.1.2])**.** *Every sequence $(G_n)_{n\in\mathbb{N}}$ of elements of a WQO $(Q, \leqslant)$ has a subsequence $\{G_{a_n}\}_{n\in\mathbb{N}}$ such that $G_{a_n} \leqslant G_{a_k}$ for every $1 \leqslant n < k$.*

2.2 (Finite) graph minors and marked graphs

Let G, H be graphs. A *minor model* of G in H is a collection of disjoint connected subgraphs $B_v, v \in V(G)$ of H, called *branch sets*, and edges $E_{uv}, uv \in E(G)$ of H, such that each E_{uv} has one end-vertex in B_u and one in B_v. We write $G < H$ to express that such a model exists, and say that G is a *minor* of H.

A *minor embedding* of G into H is a map h assigning to each $v \in V(G)$ a connected subgraph B_v of H, and to each $uv \in E(G)$ an edge E_{uv} of H such that these sets form a minor model of G in H. We write $h : G < H$ to denote that h is such a minor embedding.

A *marked graph* is a pair consisting of a graph G and a subset A of $V(G)$, called the *marked vertices*. Given two marked graphs $(G, A), (H, A')$, a *marked* *minor (model)* of G in H is defined as above, except that for each marked vertex v of G, we require that the corresponding branch set B_v contains at least one marked vertex of H. We write $(G, A) < (H, A')$, or $G <_\bullet H$ when A, A' are fixed, if this is possible. We also extend the above definition of minor embedding canonically to marked minors.

Given a set X of graphs, we write $\mathrm{Forb}(X)$ for the class of graphs H such that no element of X is a minor of H.

We recall the Robertson–Seymour Graph Minor Theorem, which we will only use in the proof Corollary 1.2 (and the warm-up Lemmas 4.1 and 4.6).

Theorem 2.3 ([15])**.** *The finite marked graphs are well-quasi-ordered under* $<_\bullet$.

We conclude this subsection with two basic facts relating connectivity and minors. The first is straightforward to prove:

Observation 2.4. *Let G, H be graphs, let $\mathcal{B}$ a minor model of G in H, and let $A \subset V(G)$. Then $\mathcal{B}$ maps each component of $G - A$ into a component of $H - \mathcal{B}(A)$.*

Here, $\mathcal{B}(A)$ stands for the image of A under $\mathcal{B}$, i.e. $\bigcup_{v \in A} B_v$.
A *block* of a graph is a maximal 2-connected subgraph.

Lemma 2.5. *Let G, H be graphs, and $\mathcal{B} = \{B_v \mid v \in V(G)\}$ a minor model of G in H. Then for each block D of G, there is a block D' of H, such that each $B_v, v \in V(D)$ intersects D'. Moreover, there is a model $\mathcal{B}'$ of D in D', obtained by intersecting each B_v with D'.*

Proof. For every $e = uv \in E(D)$, let B_e be a B_u–B_v edge in H, called a *branch edge*. Since D is 2-connected, any two edges e, f of D lie in a common cycle C. Moreover, there is a cycle C' in $\bigcup_{v \in V(C)} B_v \subseteq H$ containing B_e and B_f. Since there is a unique block containing any edge of a graph, it follows that there is a block D' of H containing $\{B_e \mid e \in E(D)\}$, and this block intersects each branch edge, and hence each branch set of D.

For the second sentence, suppose $x \in V(D')$ is a cut-vertex of H. Then all branch sets of $\mathcal{B}$ are contained in the component K of $H - x$ containing $D' - x$, except possibly for a unique branch set B containing x. If such a B exists, then after replacing it with $B \cap K$, $\mathcal{B}$ is still a model of G, because no branch edge of $\mathcal{B}$ can lie in $H - K$. Thus doing so for every cut-vertex $x \in V(D')$ we modify $\mathcal{B}$ into the desired model $\mathcal{B}'$ of G in D'. □

2.2.1 Suspensions

Given a (marked) graph G, we define its *suspension* $S(G)$ by adding an unmarked vertex s_G and joining s_G to each $v \in V(G)$ with an edge. Given a marked graph G, we define its *marked suspension* $S^\bullet(G)$ by adding a marked vertex s_G and joining s_G to each $v \in V(G)$ with an edge.

Note that $S(G)$ is always connected even if G is not. Combining this with the following observation will be often convenient, as it will allow us to assume that any bad sequences we consider consist of connected graphs.

Lemma 2.6. *Let G, H be marked graphs. Then $G < H$ if and only if $S^\bullet(G) < S^\bullet(H)$.*

Proof. The forward implication is trivial. For the backward implication, let $M = \{B_v \mid v \in V(S^\bullet(G))\}$ be a model of $S^\bullet(G)$ in $S^\bullet(H)$. If no branch set B_v contains s_H, then M witnesses that $S^\bullet(G)$, and hence G, is a minor of H and we are done. If s_H is in the branch set of s_G, then by deleting that branch set from M we obtain a model of G in H.

Thus it only remains to consider the case where some B_v with $v \in V(G)$ contains s_H. In this case B_{s_G} cannot contain s_H too, and so $B_{s_G} \subseteq H$, and B_{s_G} contains a marked vertex since s_G is marked. Then by removing B_{s_G} from M, and re-defining B_v to be B_{s_G}, we have modified M into a model of G in H. □

The unmarked version of Lemma 2.6 is also true, and easier to prove along the same lines:

Lemma 2.7. *Let G, H be (unmarked) graphs. Then $G < H$ if and only if $S(G) < S(H)$.* □

2.2.2 Minor-twins

We say that G and H are *minor-twins*, if both $G < H$ and $H < G$ hold. Any two finite minor-twins are isomorphic, but in the infinite case the relation is much more interesting.

The class of countable graphs can be decomposed into its *minor-twin classes*, whereby two graphs belong to the same class whenever they are minor-twins. The minor-twin class of a graph G will be denoted by $[G]_<$. These definitions have obvious analogues for marked minors.

Given a graph class $\mathcal{C}$, we let $|\mathcal{C}|_<$ denote the cardinality of the set of minor-twin classes of elements of $\mathcal{C}$. We define $|\mathcal{C}|_{<_\bullet}$ analogously for marked minors.

2.3 The Rank of a rayless graph

A graph is *rayless*, if it does not contain a 1-way infinite path. Schmidt [16] assigned to every rayless graph an ordinal number, its *rank*, reminiscent of the notion of rank for Borel sets. This notion often enables us to prove results about rayless graphs by transfinite induction on the rank.

The notion of rank comes from the observation that it is possible to construct all rayless graphs by a recursive, transfinite procedure, starting with the class of finite graphs and then, in each step, glueing graphs constructed in previous steps along a common finite vertex set, to obtain new rayless graphs as follows.

Definition 2.8. *For every ordinal α, we recursively define a class of graphs Rank_α by transfinite induction on α as follows:*

- Rank_0 *consists of the finite graphs; and*
- *if $\alpha > 0$, then a graph G is in Rank_α if there is a finite $S \subset V(G)$ such that each component of $G - S$ lies in Rank_β for some $\beta < \alpha$.*

Schmidt [16, 3, 5] proved that a graph is rayless if and only if it belongs to Rank_α for some α. Easily, if G is countable then so is this α, but it may be greater than ω (Observation 2.11 below). Let $\mathrm{Rank}_{<\alpha} := \bigcup_{\beta<\alpha} \mathrm{Rank}_{<\beta}$.

The *rank* $\mathrm{Rank}(G)$ of a rayless graph G is the least ordinal α such that $G \in \mathrm{Rank}_\alpha$. For a class $\mathcal{C}$ of graphs, we let $\mathrm{Rank}(\mathcal{C})$ be the least ordinal α such that $\mathrm{Rank}(G) < \alpha$ holds for every $G \in \mathcal{C}$.

Schmidt [16, 5] also proved that each rayless graph G has a unique *kernel* $A(G)$, i.e. a minimal set of vertices S such that each component of $G - S$ lies in Rank_γ for some $\gamma < \mathrm{Rank}(G)$. We claim that

For every ordinal $\alpha > 0$ every $G \in \mathrm{Rank}_\alpha$, and every $\beta < \alpha$, there are infinitely many components C of $G - A(G)$ such that $\mathrm{Rank}(C) \geqslant \beta$. (1)

Indeed, if the set $\mathcal{C}$ of such components C is finite, then the finite set $A(G) \cup \bigcup_{C \in \mathcal{C}} A(C)$ separates G into components that all have ranks less than β, yielding the contradiction $\mathrm{Rank}(G) \leqslant \beta$.

2.3.1 Rank and minors

It is well-known, and not hard to prove, that rank is monotone with respect to minors:

Observation 2.9 ([5, Proposition 4.4.]). *Let G, H be graphs with $G < H$. Then $\mathrm{Rank}(G) \leqslant \mathrm{Rank}(H)$.*

The following may be well-known but I could not find a reference:

Observation 2.10. *Let G, H be graphs with $\mathrm{Rank}(G) = \mathrm{Rank}(H) = \alpha$, and $\mathcal{B} = \{B_v \mid v \in V(G)\}$ a minor model of G in H. Then B_v intersects $A(H)$ for every $v \in A(G)$.*

Proof. Let $v \in A := A(G)$, and let $\mathcal{C} = \mathcal{C}_v$ denote the set of components of $G - A$ sending an edge to v. We claim that

$$\mathrm{Rank}(\mathcal{C}) = \alpha. \tag{2}$$

Indeed, if $\mathrm{Rank}(\mathcal{C}) < \alpha$, then $G_v := G[\{v\} \cup \bigcup \mathcal{C}]$ has rank less than α too. Moreover, each component of $G - (A - v)$ is either G_v or a component of $G - A$, and therefore it has rank less than α (Figure 1). This contradicts the fact that A is, by definition, a minimal set with this property.

Suppose B_v contains no vertex of $A' := A(H)$ for some $v \in A$. Since B_v is connected, it is contained in a component C of $H - A'$. By (2), we have $\mathrm{Rank}(G_v) = \alpha$. This remains true if we delete from G_v those elements of $\mathcal{C}_v$ containing a branch set intersecting A', since there are at most finitely many such branch sets. Let G'_v be the subgraph of G_v obtained after this deletion. Since G'_v is connected, and all its branch sets avoid A', its image under $\mathcal{B}$ is contained in C. But $\mathrm{Rank}(C) < \mathrm{Rank}(H) = \alpha$, while $\mathrm{Rank}(G'_v) = \alpha$ as observed above. This contradicts Observation 2.9, hence B_v must intersect A'. □

Observation 2.11. *For every countable ordinal α, there is a countable tree T_α with $\mathrm{Rank}(T_\alpha) = \alpha$, such that $T_\alpha < G$ for every non-empty, connected, graph G with $\mathrm{Rank}(G) \geqslant \alpha$.*

Proof. We will prove the statement by induction on α. For this it will be convenient to think of each T_α as a rooted tree, and denote its root by r_α.

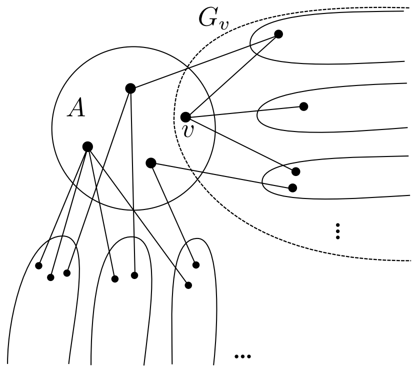

Figure 1: The components of $G - A$ and $G - (A - v)$ in the proof of (2).

Instead of $T_\alpha < G$, we will prove the following strengthening, which will be important for our induction:

for every $v \in V(G)$, there is a model of T_α in G such that the branch set of r_α contains v. (3)

For $\alpha = 0$, we just let T_α be the tree on one vertex, and note that (3) is trivially satisfied by letting $B_{r_\alpha} = \{v\}$.

For $\alpha > 0$, we construct T_α as follows. We start with the disjoint union of countably infinitely many copies of T_β for each $\beta < \alpha$, add a new vertex r_α, and join r_α to the root of each such copy of T_β with an edge. Note that $A(T_\alpha) = \{r_\alpha\}$ by construction.

This completes the construction of T_α for every ordinal α, and it now remains to prove (3), which we do by transfinite induction of α. Having checked the start $\alpha = 0$ of the induction above, we may assume that (3) holds for all ordinals $\beta < \alpha$. Given G as above and $v \in V(G)$, we construct the desired model of T_α in G as follows. Pick a v–$A(G)$ path P in G. Moreover, for every $x, y \in A(G)$, pick a x–y path P_{xy} in G. Since $A(G)$ is finite, the union of all these paths meets only a finite set $\mathcal{C}$ of components of $G - A(G)$. We let $B_{r_\alpha} = A(G) \cup \bigcup \mathcal{C}$ be the branch set of r_α in our model. All other branch sets will be chosen within $G - \bigcup \mathcal{C}$.

Let $(C_n)_{n \in \mathbb{N}}$ be an enumeration of the components of $T_\alpha - r_\alpha$, and recall that each C_n is isomorphic to T_β for some $\beta < \alpha$. For $i = 1, 2, \ldots$, we recursively find a model of C_i in $G - \bigcup \mathcal{C}$ as follows. We pick a component C'_i of $G - \bigcup \mathcal{C}$ with $\mathrm{Rank}(C'_i) \geqslant \beta$ such that $C'_i \notin \mathcal{C}$, and $C'_i \neq C'_j$ for any $j < i$, which C'_i exists by (1). Since G is connected, there is a vertex $v_i \in C'_i$ sending an edge e_i to $A(G)$. Applying the inductive hypothesis (3) with G replaced by C'_i, and v replaced by v_i, and α replaced by β, we obtain a minor model of $C_i \cong T_\beta$ in C'_i, in which the branch set corresponding to r_β contains v_i. Adding the edges e_i, and the branch set B_{r_α} to these models for all $i \in \mathbb{N}$ we obtain the desired model of T_α in G. □

2.4 Better-quasi-orders

Rather than repeating the original definition of a better-quasi-order, we will work with an equivalent one. Intuitively, a quasi-order $(Q, \leqslant)$ is better-quasi-ordered if it is well-quasi-ordered, and so are its subsets, sets of subsets, sets of sets of subsets, and so on transfinitely, whereby we recursively extend $\leqslant$ from elements of Q to subsets of Q. To make this precise, we need the following terminology.

Let $\mathcal{P}^*(A)$ denote the set of non-empty subsets of a set A, i.e. $\mathcal{P}^*(A) := \mathcal{P}(A)\backslash\{\varnothing\}$. Let Q be a quasi-order. For every ordinal α we define, by transfinite induction, the 'iterated power set' $V^*_\alpha(Q)$ as follows. We start our induction by setting $V^*_0(Q) := Q$. Having defined $V^*_\alpha(Q)$, we let $V^*_{\alpha+1}(Q) := \mathcal{P}^*(V^*_\alpha(Q))$. Finally, if α is a limit ordinal, we let $V^*_\alpha(Q) := \bigcup_{\beta<\alpha} V^*_\beta(Q)$. We say that $X \in V^*_\alpha(Q)$ is *hereditarily countable* if it is a countable set of hereditarily countable sets (the latter having been defined recursively, starting by declaring each element of Q to be hereditarily countable).

Having defined $V^*_\alpha(Q)$ for every α, we let $V^*(Q) := \bigcup_\alpha V^*_\alpha(Q)$. The *$Q$-rank* $Q\text{Rank}(X)$ of an element $X \in V^*_\alpha(Q)$ is defined as $\alpha + 1$ where α is the least ordinal such that $X \in V^*_\alpha(Q)$. Thus $Q\text{Rank}(X) = 1$ if and only if $X \in Q$ (the $+1$ may look strange for now, but it will be justified later).

Definition 2.12. *Define a quasi-order $\leqslant$ on $V^*(Q)$ as follows*

(i) if $X, Y \in Q$, then $X \leqslant Y$ in $V^*(Q)$ if and only if $X \leqslant Y$ in Q;

(ii) if $X \in Q$ and $Y \notin Q$, then $X \leqslant Y$ if and only if there exists $Y' \in Y$ with $X \leqslant Y'$;

(iii) if $X \notin Q$ and $Y \notin Q$, then $X \leqslant Y$ if and only if for every $X' \in X$ there exists $Y' \in Y$ with $X' \leqslant Y'$.

Let $H^*_{\omega_1}(Q)$ denote the set of hereditarily countable elements of $V^*_{\omega_1}(Q)$ equipped with the above quasi-order induced from $V^*(Q)$.

Theorem 2.13 ([10, Theorem 3.45])**.** *A quasi-order Q is BQO if and only if $H^*_{\omega_1}(Q)$ is WQO.*

We will effectively use Theorem 2.13 as our definition of *better-quasi-ordering*.

Let $Seq(Q)$ be the set of all finite or countably infinite sequences with elements in $(Q, \leqslant)$. We endow $Seq(Q)$ by a quasi-ordering $\preceq$ by letting $S \preceq T$ if there exists an embedding from F into G, i.e. a strictly increasing map ϕ from the index set of S to that of T, such that $S(i) \leqslant T(\phi(i))$ for every i. We will use the following well-known lemmas about better-quasi-ordering.

Lemma 2.14 ([8], [7, Lemma 4])**.** *If a quasi-order Q is BQO, then so is $Seq(Q)$.*

Lemma 2.15 ([7, Lemma 3])**.** *If two quasi-orders Q_1, Q_2 are BQO, then so is $Q_1 \times Q_2$.*

3 From WQO to BQO

The aim of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. Let Q be the set of finite *1-connected graphs*, i.e. connected graphs with at least two vertices. By Lemmas 2.14 and 2.15, it suffices to prove that Q is better-quasi-ordered. Indeed, any finite graph G is the disjoint union of a number n_G of isolated vertices and a sequence s_G of 1-connected subgraphs. Thus we can represent G as a pair (n_G, s_G), and apply Lemma 2.15 to such pairs after applying Lemma 2.14 to these sequences, recalling that $\mathbb{N}$ is well-ordered and hence better-quasi-ordered.

It thus remains to prove

$$\text{If the class } \mathcal{R} \text{ of countable rayless graphs is WQO, then } Q \text{ is BQO.} \tag{4}$$

To prove this we will define a map T that assigns to each set X in $H^*_{\omega_1}(Q)$ a rayless graph $T(X)$ in such a way that

$$T(X) < T(Y) \text{ implies } X \leqslant Y \text{ for every } X, Y \in H^*_{\omega_1}(Q), \tag{5}$$

where $\leqslant$ stands for the relation of Definition 2.12. To see how this implies 4, recall that, by Lemma 2.13, if $H^* := H^*_{\omega_1}(Q)$ is WQO then Q is BQO. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of elements of H^*. If the countable rayless graphs are WQO, then $\{T(X_n)\}_{n\in\mathbb{N}}$ is good, hence so is $(X_n)_{n\in\mathbb{N}}$ by 5, and so H^* is WQO as desired.

It remains to define T and prove that it satisfies 5. We define $T(X), X \in H^*$ by transfinite induction on the Q-rank of X as follows. If $Q\text{Rank}(X) = 1$, in which case $X \in Q$ is a finite connected graph (on at least two vertices), then $T(X)$ comprises a countably infinite collection of pairwise disjoint copies of X, and an additional vertex r, called the *root*, joined by an edge to all other vertices.

If $Q\text{Rank}(X) > 1$, then $T(X)$ comprises a countably infinite collection of pairwise disjoint copies of $T(X')$ for each $X' \in X$, and an additional root vertex r joined by an edge to the root of each such $T(X')$.

It is easy to show, by transfinite induction on $Q\text{Rank}(X)$, that $T(X)$ is a countable graph, and that it is rayless; to see the latter, notice that any ray in $T(X)$ can visit r at most once, hence it would need to have a sub-ray in a copy of $T(X')$ for some $X' \in X$ if $Q\text{Rank}(X) > 1$, or in a copy of X if $Q\text{Rank}(X) = 1$. It is not hard to show, again by transfinite induction on $Q\text{Rank}(X)$, that

$$\text{Rank}(T(X)) = Q\text{Rank}(X) \text{ for every } X \in H^*. \tag{6}$$

(This justifies the $+1$ in the definition of $Q\text{Rank}(X)$; without it this equation would hold only when $Q\text{Rank}(X) \geqslant \omega$.)

We call $r =: r(T(X))$ the *root* of $T(X)$. The *children* of r are the copies of the roots $\{r(T(X')) \mid X' \in X\}$. The *descendant* relation is the transitive closure of the child relation just defined. Given a descendant r' of r, we write $\lceil r' \rceil$ for the component of $T(X)$ containing r' formed when deleting the edge from r' to its parent. We set $\lceil r \rceil = T(X)$. Notice that $\lceil r' \rceil$ is isomorphic to $T(Y)$ for some element Y of the transitive closure of X.

By construction, we can assign to each vertex v of $T(X), X \in H^*$ an ordinal number called the *level* of v, similar to the notion of QRank, as follows. If $X \in Q$, each vertex of X and its copies has level 0. For $X \in H^*$, we proceed inductively to define the *level* of $r(T(X))$ to be the smallest ordinal that is larger than the level of each vertex $v \neq r(T(X))$ of $T(X)$, which is defined in a previous step of the induction since $v \in V(T(X'))$ for some $X' \in X$.

Notice that for each triangle Δ of $T(X)$, at least two of the vertices of Δ have level 0. Recall moreover that each $X \in Q$ contains at least one edge. It thus follows from our construction that, for every $X \in H^*$,

a vertex v of $T(X)$ lies in infinitely many triangles of $T(X)$ if and only if v has level 1, and no vertex of level > 1 lies in a triangle. (7)

We now prove that this definition of T satisfies 5, by a nested transfinite induction on $Q\mathrm{Rank}(X)$ and $Q\mathrm{Rank}(Y)$.

Remark 3.0.1. More generally, we can let Q' be any non-empty family of 1-connected finite graphs. Then 7 still holds. By restricting (5) to Q', we deduce the following variant of (4): if $T[Q']$, i.e. the image of $H^*_{\omega_1}(Q')$ under T, is well-quasi-ordered, then Q' is better-quasi-ordered. The remainder of this proof works verbatim when replacing Q by Q', and $\mathcal{R}$ by $T[Q']$ in (4).

Our inductive proof of (5) starts with $Q\mathrm{Rank}(X)$ being 1: assume that $T(X) < T(Y)$ holds for some $X \in Q$ and $Y \in H^*$. Let B_r denote the branch set corresponding to the root $r = r(T(X))$ in some minor model $\mathcal{B}$ of $T(X)$ in $T(Y)$. We claim that B_r contains at least one vertex of level 1 in Y. Indeed, by (7) r lies in infinitely many triangles of $T(X)$, and if B_r avoids level 1 vertices of $T(Y)$, then it lies in at most finitely many triangles of any minor of $T(Y)$.

Let G be one of the copies of X in $T(X)$, let xy be an edge of G, and notice that xyr is a triangle of $T(X)$. Then the branch sets B_x, B_y of $\mathcal{B}$ are contained in a component C of level 0 vertices of $T(Y)$. Let r' be the unique level 1 vertex of $T(Y)$ sending edges to C. Notice that $r' \in B_r$. Since G is connected, and r' separates C from the rest of $T(Y)$, we deduce that $B_v \subset C$ for every $v \in V(G)$. By our construction of $T(Y)$, there are infinitely many level 0 components C' of $T(Y)$ isomorphic with C and incident with r'. It follows that

$$T(X) \leqslant \lceil r' \rceil. \tag{8}$$

If $Q\mathrm{Rank}(Y) = 1$, then C is just a copy of Y and we obtain the desired $X \leqslant Y$ since G was a copy of X. If $Q\mathrm{Rank}(Y) > 1$, then let Y' be the element of Y such that some copy of $T(Y')$ in $T(Y)$ contains r', and hence $\lceil r' \rceil$. In this case (8) implies $T(X) \leqslant T(Y')$, and by transfinite induction on $Q\mathrm{Rank}(Y)$, of which the previous case is the initial step, we deduce $X \leqslant Y'$, and hence $X \leqslant Y$ by (ii) of Definition 2.12.

Thus we have completed the initial step $Q\mathrm{Rank}(X) = 1$ of our inductive proof of (5). Assume now that $Q\mathrm{Rank}(X) > 1$, and $T(X) \leqslant T(Y)$ holds for some $Y \in H^*$. We cannot have $Q\mathrm{Rank}(Y) = 1$ by Observation 2.9 and (6). Thus we are in case (iii) of Definition 2.12, and so our task is to find, for each $X' \in X$, some $Y' \in Y$ such that $X' \leqslant Y'$.

To this aim, let again B_r denote the branch set of some minor model $\mathcal{B}$ of $T(X)$ in $T(Y)$ corresponding to the root $r = r(T(X))$. Let r' be the vertex of B_r of maximal level among all vertices of B_r; this exists because B_r is connected, and hence if it contains vertices of all levels $\beta < \alpha$ for some ordinal α, then it must also contain a vertex of level α. By (7), the level of r' is at least 1. Notice that all but at most one of the edges of r are mapped by $\mathcal{B}$ to descendants of r', because r' sends at most one edge to a non-descendant. Call this edge e if it exists. Since $T(X)$ contains infinitely many pairwise disjoint copies of $T(X')$

incident with r, at least one (in fact almost all) of these copies G is mapped by $\mathcal{B}$ to a subgraph of $T(Y)$ avoiding e. Therefore, G is mapped by $\mathcal{B}$ into $\lceil r'' \rceil$ for some child r'' of r', because G is connected, r' separates its children, and $r' \in B_r$ cannot lie in B_v for any $v \in V(G)$. Let Y' be the element of Y for which some copy of $T(Y')$ contains r'', which exists since $r'' \neq r(T(Y))$ as r'' is a child of another vertex. Then $T(Y')$ contains $\lceil r'' \rceil$, and hence $T(X') < T(Y')$ because $T(X') < \lceil r'' \rceil$. Since $Q\text{Rank}(X') < Q\text{Rank}(X)$, our inductive hypothesis yields $X' \leqslant Y'$ as desired, completing the inductive step. □

By Remark 3.0.1, we immediately deduce

Corollary 3.1. *Let F be a set of finite graphs, and let Q be the set of 1-connected elements of F. If $T[Q']$ is well-quasi-ordered, then F is better-quasi-ordered.*

It is not hard to prove the converse of (5), by induction on the Q-rank by recursively preserving the property that the branch set of the root contains the root of the target graph:

$$X \leqslant Y \text{ implies } T(X) < T(Y) \text{ for every } X, Y \in H^*_{\omega_1}(Q). \tag{9}$$

Thus combining (5), (9) and Theorem 2.13, and recalling that $T[Q]$ denotes the image of $H^*_{\omega_1}(Q)$ under T, we deduce

The finite graphs are better-quasi-ordered if and only if $T[Q]$ is well-quasi-ordered. (10)

4 Warm-up: Graphs of rank 1

This section introduces some of the fundamental techniques used throughout the paper, and serves as a preparation towards the more difficult Section 5. It handles graphs of rank 1. The section culminates with the proof of Theorem 1.2, which is based on the same techniques.

Let UF (to be read "union-finite") denote the class of countable graphs G such that each component of G is finite. Note that $UF \subset \text{Rank}_1$. For $G \in UF$, we let $\mathcal{C}(G)$ denote the class of finite graphs H such that $H < G$. The following is a toy version of Lemma 5.1, a central tool for the proof of Theorem 1.3.

Lemma 4.1. *For every $G, G' \in UF$, we have $G < G'$ if and only if $\mathcal{C}(G) \subseteq \mathcal{C}(G')$.*

To see the relevance of this to cardinality of minor-twin classes as in Theorem 1.3 (d), let us prove that Lemma 4.1 implies that $|UF|_< = \aleph_0$. In fact, this statement is equivalent to the Graph Minor Theorem:

Corollary 4.2. *$|UF|_< = \aleph_0$ if and only if the finite graphs are well-quasi-ordered.*

Proof. For the backward direction, note that Lemma 4.1 says that the minor-twin class $[G]_<$ of $G \in UF$ is determined by $\mathcal{C}(G)$. Using the fact that the finite graphs are well-quasi-ordered, we can express $\mathcal{C}(G)$ as $\mathcal{C}(G) = \text{Forb}(X)$ for a finite set X of finite graphs. Thus there are countably many choices for $\mathcal{C}(G)$, hence for $[G]_<$, since there are countably many finite graphs to choose X from.

For the forward direction, suppose for a contradiction that $(G_n)_{n\in\mathbb{N}}$ is an anti-chain of finite graphs under the minor relation. By Lemma 2.7, we may assume that each G_n is connected. For each $X \subset \mathbb{N}$, let $\mathcal{C}_X$ be the (minor-closed) class of finite graphs $\mathrm{Forb}(X)$. Let $G_X := \bigcup \mathcal{C}_X \in UF$ be the (countably infinite) graph obtained as the disjoint union of all the graphs in $\mathcal{C}_X$. Note that for every finite graph H, we have $H < G_X$ if and only if $H \in \mathcal{C}_X$ because each $G_i \in X$ is connected. This implies that G_X is not a minor-twin of G_Y whenever $X \neq Y$, because any graph in the symmetric difference $X \triangle Y$ is a minor of exactly one of G_X, G_Y. Since there are continuum many $X \subset \mathbb{N}$, we have obtained continuum many minor-twin classes $[G_X]_<$ (thus we could add $|UF|_< < 2^{\aleph_0}$ as a further equivalent statement). □

Generalising the idea of the proof of Corollary 4.2 to higher ranks will be the key to proving the equivalence (a) $\leftrightarrow$ (d) of Theorem 1.3, the main difficulty being that we do not have an analogue of the Graph Minor Theorem for ranks higher than 0.

We prepare the proof of Lemma 4.1 by recalling a well-known idea:

Hilbert's Hotel Principle: Suppose a hotel has infinitely many single rooms, numbered $R_1, R_2, \ldots$, and each R_i is occupied by a guest G_i. If a new guest G arrives, they can be accommodated in R_1, by moving each G_i to G_{i+1}.

Proof of Lemma 4.1. The forward implication follows immediately by restricting a minor model of G in G' to any $H \in \mathcal{C}(G)$.

For the backward implication, suppose $\mathcal{C}(G) \subseteq \mathcal{C}(G')$, and let G_n be the union of the first n components of G in a fixed but arbitrary enumeration of its components. Let G'_n be a subgraph of G' such that $G_n < G'_n$, which exists since $\mathcal{C}(G) \subseteq \mathcal{C}(G')$. Easily, we may assume G'_n is finite. Let $h_n : G_n < G'_n$ be a minor embedding (as defined in Section 2.2).

Call a component C_i of G *h-stable*, if $\bigcup_n h_n(C_i)$ is finite; in other words, if C_i is mapped to a finite set of components of G' by the $h_n, n \in \mathbb{N}$. Let $(S_n)_{n\in\mathbb{N}}$ be an enumeration of the h-stable components of G, and $(U_n)_{n\in\mathbb{N}}$ an enumeration of the other components of G. One of these enumerations may be finite, or even empty.

Let G_S denote the (possibly empty) subgraph of G consisting of its h-stable components. By a standard compactness argument, there is a minor embedding $h_S : G_S < G'$ such that $h_S(S_i)$ coincides with $h_n(S_i)$ for infinitely many values of n. If $G_S = G$ we are done, so suppose from now on U_1 exists.

We will now modify h_S, recursively in at most ω steps $i = 1, 2, \ldots$, into a minor embedding of G into G', whereby in step i we handle U_i. Importantly, the image of h_S might be all of G', and so we may have to reshuffle G_S inside G' to make space for the U_i's.

We set $h^0_S := h_S$, and assume recursively that $h^j_S : G_S \cup \{U_1, \ldots, U_{j-1}\}$ has been defined for every $j < i$. Moreover, we assume that a finite number of components of G have been *nailed* into components of G', which means that we promise that h^i_S will coincide with h^{i-1}_S on all components that have been nailed before. We will ensure that every component of G—stable or not— will be nailed in some step. No components have been nailed at the beginning of step 1. If U_i does not exist, then we just let $h^i_S = h^{i-1}_S$, and nail S_i —this is the easy case, and we can just terminate the process as h^{i-1}_S is a minor embedding of G into G' in this case.

Otherwise, let $(C^i_n)_{n \in \mathbb{N}}$ be an infinite sequence of distinct components of G' into which U_i is embedded by some h_n, which exists since U_i is not h-stable. As the finite graphs are well-quasi-ordered by Theorem 2.3, $(C^i_n)_{n \in \mathbb{N}}$ has an infinite subsequence $(Y_n)_{n \in \mathbb{N}}$ such that $Y_r < Y_m$ for every $r < m \in \mathbb{N}$ by Observation 2.2. Pick $k = k(i)$ such that no $Y_m, m \geqslant k$ has been nailed yet.

If Y_k does not intersect the image of h^{i-1}_S, we let h^i_S extend h^{i-1}_S by embedding U_i into Y_k; this is possible since some h_n embeds U_i into Y_k by the definition of the latter. We nail U_i to Y_k (thereby promising that h^j_S will embed U_i into Y_k for every $j \geqslant i$). Finally, if S_i exists, we nail it to the component containing $h^{i-1}_S(S_i)$, again promising that $h^j_S(S_i)$ is fixed from now on.

It remains to consider the —more difficult— case where Y_k intersects the image of h^{i-1}_S. In this case, imitating Hilbert's Hotel principle, we modify h^{i-1}_S into h^i_S by shifting the 'contents' of each $Y_m, m \geqslant k$ to Y_{m+1}, and mapping U_i into Y_k. To make this precise, fix a minor embedding $g_m : Y_m < Y_{m+1}$ for every $m \geqslant k$, which exists by the definition of $(Y_n)_{n \in \mathbb{N}}$. Then, for every $m \geqslant k$, and every component C of G such that $h^{i-1}_S(C)$ intersects Y_m —and therefore $h^{i-1}_S(C)$ is contained in Y_m— we let $h^i_S(C) := g_m \circ h^{i-1}_S(C)$, so that h^i_S embedds C into Y_{m+1}. Thus h^i_S now maps the domain $G_S \cup \{U_1, \ldots, U_{i-1}\}$ of h^{i-1}_S to $G' \backslash Y_k$. We extend h^i_S to U_i, embedding U_i into Y_k (by imitating some h_n), and we nail U_i to Y_k.

Again, if S_i exists, we nail it to the component containing $h^i_S(S_i)$ —which may coincide with the component containing $h^{i-1}_S(S_i)$, or have been shifted from some Y_m to Y_{m+1}.

This completes the definition of $h^i_S, i \in \mathbb{N}$. Note that each of U_i, S_i that exists has been nailed by step i, and its h^ℓ_S-image is fixed for $\ell \geqslant i$. Thus h^i_S converges, as $i \to \infty$, to a minor embedding $h : G \to G'$, proving our claim $G < G'$. □

Our next result extends Lemma 4.1 from UF to its superclass Rank_1. For this we will need to adapt the above definition of $\mathcal{C}(G)$ (Definition 4.5 below), whereby we will have to consider marked graphs.

Definition 4.3. *For $G \in \text{Rank}_\alpha, \alpha \geqslant 1$, a* co-part *of G is a component of $G - A(G)$. Given a co-part C of G, we call the subgraph $G[C \cup A(G)]$ induced by $C \cup A(G)$ a* part *of G.*

Note that each part of G has lower rank than that of G.

Definition 4.4. *Let $\text{Rank}^\bullet_\alpha$ denote the class of marked graphs (G, M) with $G \in \text{Rank}_\alpha$ and M finite. Define $\text{Rank}^\bullet_{<\alpha}$ analogously.*

Definition 4.5. *Given a rayless graph $G \in \text{Rank}_\alpha$, we let $\mathcal{C}^\bullet(G)$ denote the class of marked graphs in $\text{Rank}^\bullet_{<\alpha}$ that are marked-minors of $(G, A(G))$.*

The following lemma, which extends Lemma 4.1, is again not formally needed for our later proofs; we include it as a warm-up towards the more difficult Lemma 5.1, but the reader will need to be familiar with its proof.

Lemma 4.6. *Let $G, G' \in \text{Rank}_1$, and suppose $|A(G)| = |A(G')|$. We have $G < G'$ if and only if $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(G')$.*

Proof. As before, the forward implication is straightforward.

For the backward implication, let $(P_n)_{n\in\mathbb{N}}$ be an enumeration of the parts of G, and $(P'_n)_{n\in\mathbb{N}}$ an enumeration of the parts of G'. Let G_n be the graph $\bigcup_{i\leqslant n} P_i$ with $A := A(G)$ marked. Choose a (marked) minor embedding $h_n : G_n < G'_n$, where $G'_n \subset G'$ is finite and has $A' := A(G')$ as its marked vertex set. Choose h_n so as to minimize the number of vertices of $A(G')$ in the image of h_n.

In this case, we will use $(h_n)_{n\in\mathbb{N}}$ to construct a model of G in G' using the ideas of the proof of Lemma 4.1, whereby we need to pay special attention to the vertices in A, in particular to how their branch sets intersect $G'\backslash A'$.

Since each co-part $P_i\backslash A$ is connected, it is mapped to a co-part of G' by each $h_n, n \geqslant i$. Note that $h_n(x) \cap A'$ is a singleton $\{x'\}$ for every $x \in A$ and $n \in \mathbb{N}$ by Observation 2.10. Thus this map $x \mapsto x'$ is a bijection from A to A'. By passing to a subsequence of $\{h_n\}$ if necessary, we may assume that this bijection is fixed for every $n \in \mathbb{N}$. Moreover, we can choose subsequences $\{h_n\} \supseteq \{h^1_n\} \supseteq \{h^2_n\} \dots$ of $\{h_n\}$ such that for every $x \in A$ and every $i \in \mathbb{N}$, the intersection of the branch set $h^i_n(x)$ with the parts $\{P'_1, \dots, P'_i\}$ is independent of n; indeed, having chosen $\{h^{i-1}_n\}$, we observe that since P'_i and A are finite, there is an infinite subsequence $\{h^i_n\}$ along which $h^{i-1}_n(x) \cap P'_i$ is constant for every $x \in A$.

Let $h'_n := h^n_n$. Note that $\{h'_n\}$ is a subsequence of $\{h_n\}$, and that $h'_n(x)$ converges for every $x \in A$, to a connected subgraph $h_A(x)$ of G' containing x' and no other vertex of A'. This is the beginning of our construction of a minor model of G in G'. Let us now embed the vertices in $G\backslash A$.

Similarly to the proof of Lemma 4.1, we call a part P_i of G *h'-stable*, if $\bigcup_n h'(P_i)$ is finite. Let $(S_n)_{n\in\mathbb{N}}$ be an enumeration of the h'-stable parts of G, and $(U_n)_{n\in\mathbb{N}}$ an enumeration of its other parts. Let $G_S := \bigcup_{n\in\mathbb{N}} S_n$. Again, a standard compactness argument yields a minor embedding $h_S : G_S < G'$ such that $h_S(S_i\backslash A)$ coincides with $h'_n(S_i\backslash A)$ for infinitely many values of n whenever S_i exists. Note that h_S extends h_A since the latter is the limit of the restriction of $\{h'_n\}$ to A. By construction, $h_S =: h^0_S$ is a minor embedding of G_S into G'.

We continue by following the lines of the proof of Lemma 4.1: for $i = 1, 2, \dots$, if U_i exists, we let $\{C^i_n, n \in \mathbb{N}\}$ be an infinite sequence of distinct parts of G' such that each C^i_n contains $h'_m(U_i\backslash A)$ for some $m \in \mathbb{N}$, whereby we used the fact that each h_m maps each co-part of G to one of G'.

Combining the marked-graph version of the Graph Minor Theorem 2.3 with Observation 2.2, we deduce that $(C^i_n)_{n\in\mathbb{N}}$ has an infinite subsequence $(Y_n)_{n\in\mathbb{N}}$ such that $Y_r < Y_m$ for every $r < m \in \mathbb{N}$.

From now on we will not need to use the assumption that $\mathrm{Rank}(G) = \mathrm{Rank}(G') = 1$; this will be important later, as the rest of this proof is also used for Lemma 5.1.

As before, we pick $k = k(i)$ such that no $Y_m, m \geqslant k$ has been nailed yet, and the interesting case is where Y_k intersects the image of h^{i-1}_S. In this case, we want to apply Hilbert's Hotel principle again to 'shift' the h^{i-1}_S-image within each $Y_m, m \geqslant k$ to Y_{m+1}, but we need to be careful with the image of A. For this, we start by picking a marked-minor embedding $g_m : Y_m < Y_{m+1}$ for every $m \geqslant k$. Note that as A' is the set of marked vertices of each Y_n, each g_m induces a permutation π_m of A'. Moreover, by composing consecutive g_m's we obtain marked-minor embeddings $g_{mt} : Y_m < Y_t$ for every $t \geqslant m \geqslant k$, which again induce permutations π_{mt} on A'. Applying Lemma 4.8 we find a subsequence (Y'_n) of (Y_n) such that each of the corresponding permutations π_{mt} is the identity. We may assume without loss of generality that $Y' = Y$. Thus we can now repeat the idea of Lemma 4.1 to shift the h_S^{i-1}-image within each $Y_m, m \geqslant k$ to Y_{m+1} and embed, and nail, U_i to Y_k to obtain h_S^i. We also nail S_i, if it exists, to the part of G' containing $h_S^i(S_i \backslash A)$. As before, the h_S^i converge as $i \to \infty$ to a minor embedding of G in G'. □

4.1 Proof of Corollary 1.2

We now use the above techniques to prove Seymour's self-minor conjecture for rayless graphs (Corollary 1.2), which we restate here for convenience:

Corollary 4.7. *For every infinite rayless graph G, there is a proper minor embedding $g : G < G$.*

We start with a simple lemma about permutations, which we will apply to permutations of $A(G)$ arising from self-minor models of G.

Lemma 4.8. *For every sequence $(\pi_n)_{n \in \mathbb{N}}$ of permutations of a finite set A there is an infinite index set $Y \subseteq \mathbb{N}$ such that $\pi_{jk} = Id$ for every $j < k \in Y$, where $\pi_{jk} := \pi_{k-1} \circ \pi_{k-2} \circ \ldots \circ \pi_{j+1} \circ \pi_j$.*

Proof. For every $k \in \mathbb{N}$ let $S_k := \pi_{k-1} \circ \pi_{k-2} \circ \ldots \circ \pi_0$, let π be a permutation of A that coincides with S_k for infinitely many k, and let Y be the set of those k except the least one. □

Proof of Corollary 4.7. We can find a subgraph $H \subset G$ of rank 1 as follows. If $\mathrm{Rank}(G) = 1$ we just let $H := G =: G_0$. If $\mathrm{Rank}(G) > 1$, we pick a co-part G_1 of G with $\mathrm{Rank}(G_1) \geqslant 1$; such a G_1 exists, because if all co-parts of G have rank 0 then G has rank 1 by the definitions. We then iterate, with G replaced by G_1, to obtain a sequence $G_1, G_2, \ldots$ of subgraphs of G, each of rank at least 1. Note that $\mathrm{Rank}(G_i) > \mathrm{Rank}(G_{i+1})$ since G_{i+1} is a co-part of G_i. Thus the sequence terminates because the ordinal numbers are well-ordered, and we let H be the final member G_k of this sequence. Clearly, $\mathrm{Rank}(H) = 1$, for the sequence would have continued otherwise.

Let $A'(H) := A(G) \cup A(G_1) \ldots \cup A(G_k)$. (We can think of $A'(H)$ as the union of $A(H)$ with all its 'parent' vertices.) Note that $A'(H)$ is finite. Let $P_i, i \in \mathcal{I}$ be a (possibly transfinite) enumeration of the parts of H, and let $P_i' := G[A'(H) \cup P_i]$. Apply Theorem 2.3 to $(P_n')_{n \in \mathbb{N}}$ as above (with $A'(H)$ always marked) to find an infinite $<$-chain that fixes $A'(H)$, for which we also use Lemma 4.8 below, and then apply the HH principle to form a proper self-minor model of $G[A'(H) \cup \bigcup_{i \in \omega} P_i]$ (note that we are ignoring P_i for $i \geqslant \omega$ so far). Extend this model to $G - H$, and to $P_i, i \geqslant \omega$, by the identity map, to obtain the desired $g : G < G$. □

4.2 A toy extension to higher rank

Generalising Lemmas 4.1 and Lemma 4.6 to higher ranks is much harder in general (and the topic of Section 5), but there is a class of graphs for which it becomes substantially easier. This subsection, which can be skipped, is about this class.

We let $\mathcal{C}^*(G)$ denote the class of marked graphs in $\mathrm{Rank}_{<\alpha}$ consisting of finite unions of parts H of $G - A(G)$, with $A(G)$ marked. We say that G is *ω-repetitive*, if for each of its parts H, there are infinitely many parts of G isomorphic to H.

Lemma 4.9. *For every $\alpha < \omega_1$ and ω-repetitive graphs $G, G' \in \mathrm{Rank}_\alpha$, we have $G < G'$ if and only if $\mathcal{C}^*(G) \subseteq \mathcal{C}^*(G')$.*

Proof. The forward implication follows again immediately by restricting a minor model of G in G' to any $H \in \mathcal{C}^*(G)$.

For the backward implication, let $(P_n)_{n\in\mathbb{N}}$ be an enumeration of the parts of G, and let $G_n := \bigcup_{i \leqslant n} P_i$.

Since G' is ω-repetitive, we can *dedicate* each part of G' to a unique part of G so that for each P_i there are infinitely many parts of G' of each isomorphism class that are dedicated to P_i. Then, for each G_n, we can choose a minor embedding $f_n : G_n < G'$ that embeds each $P_i, i \leqslant n$ to the union of $A(G')$ with parts of G' dedicated to P_i.

Since $A(G')$ is finite, there is an infinite subsequence $(f^1_n)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f^1_j(P_1) \cap A(G')$ is the same for every $j \in \mathbb{N}$. We may assume that the $f^1_j(P_1)$ coincide outside $A(G')$ too, because P_1 is always mapped to parts of G' dedicated to it. Repeating this argument with $P_2, P_3, \ldots$, we obtain subsequences $f^2 \supset f^3, \ldots$, such that $\bigcup_n f^n_1(P_n)$ is a minor embedding of G into G'. □

5 Extending Lemma 4.1 to Rank > 1

Recall that $\mathcal{C}^\bullet(G)$ denotes the class of marked minors of $(G, A(G))$ of lower rank (Definition 4.5). The following is a key lemma for the proof of Theorem 1.3, and it generalises Lemma 4.6.

Lemma 5.1. *Let G, H be countable graphs with $\mathrm{Rank}(H) = \mathrm{Rank}(G) \leqslant \alpha < \omega_1$, and $|A(G)| = |A(H)|$. Assume $\mathcal{C}^\bullet(H)$ is well-quasi-ordered, and $|\mathrm{Rank}^\bullet_\beta \cap \mathcal{C}^\bullet(G)|_{<_\bullet}$ is countable for every $\beta < \alpha$. Then we have $G < H$ if and only if $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(H)$.*

Compared to Lemmas 4.1 and 4.6, this statement imposes two additional conditions in order to be able to handle ranks $\alpha > 1$. To appreciate the role of these conditions, recall that when we used Lemma 4.1 to prove Corollary 4.2, we used the Graph Minor Theorem and the fact that there are countably many isomorphism types of finite graphs. We do not have analogous statements for higher ranks, and so our condition that $\mathcal{C}^\bullet(H)$ is well-quasi-ordered replaces the former, and the condition $|\mathrm{Rank}^\bullet_\beta \cap \mathcal{C}^\bullet(G)|_{<_\bullet} = \aleph_0$ replaces the latter. Note that both conditions are about graphs of lower rank than that of G, H, which will allow us to apply Lemma 5.1 within inductive arguments.

Lemma 5.1 is an important reason why we are forced to consider marked graphs even though we are mainly interested in unmarked ones:

Remark 5.0.1. We cannot replace $\mathcal{C}^\bullet(G)$ in Lemma 5.1 by its unmarked version $\mathcal{C}(G)$, as shown by the following example. Let G_0 be the disjoint union of the finite cliques $K_n, n \in \mathbb{N}$. Let $H = S(G_0)$ be its suspension (as defined in Section 2.2.1, and $G := S(H)$. Easily, every finite minor of G is a minor of H, and nevertheless $G \nless H$ by Observation 2.10 since $|A(G)| = 2$ and $|A(H)| = 1$. This example also shows that it is important to mark the apex vertices and only those in the definition of $\mathcal{C}^\bullet(G)$.

Remark 5.0.2. We can also not replace $\mathcal{C}^\bullet(G)$ by its finitary version $\mathcal{C}^\bullet_{\mathrm{fin}}(G)$, i.e. the class of finite marked graphs that are marked-minors of $(G, A(G))$, as shown by the following example. Let $H = S(G_0)$ be as above, and let $H' = S(\omega \cdot H)$ be the suspension over the disjoint union of ω copies of H (thus $\mathrm{Rank}(H') = 2$). For every $n \in \mathbb{N}$, let $S_n := S(\omega \cdot K_n)$. Let $G' := S(\bigcup_{n\in\mathbb{N}} S_n)$ (again $\mathrm{Rank}(G') = 2$). Note that $H \nless S_n$ for any n, and using this it is not hard to see that $G' \nless H'$. On the other hand, both $\mathcal{C}^\bullet_{\mathrm{fin}}(G'), \mathcal{C}^\bullet_{\mathrm{fin}}(H')$ consist of all finite graphs with at most one marked vertex.

We prepare the proof of Lemma 5.1 with two lemmas:

Lemma 5.2. *Suppose* $\mathrm{Rank}(G) = \alpha < \omega_1$*, and* $|\mathrm{Rank}^\bullet_\beta \cap \mathcal{C}^\bullet(G)|_{<_\bullet}$ *is countable for every* $\beta < \alpha$*. Then there is a sequence* $G_1 \subset G_2 \subset \ldots$ *of subgraphs of* G*, containing* $A := A(G)$*, such that*

(i) $\mathrm{Rank}(G_i) < \mathrm{Rank}(G)$ *for every* $i \in \mathbb{N}$*; and*

(ii) for every $G' \subset G$ *containing* $A(G)$ *with* $\mathrm{Rank}(G') < \mathrm{Rank}(G)$ *there is* n *such that* $(G', A) <_\bullet (G_n, A)$*.*

Proof. By our assumption, the family of marked graphs

$$\{(G', A) \mid A \subset G' \subset G, \mathrm{Rank}(G') < \mathrm{Rank}(G)\} \subseteq \mathcal{C}^\bullet(G)$$

decomposes into countably many marked-minor-twin classes (whereby we use the fact that there are countably many ordinals $\beta < \alpha$). Enumerate these classes as $(\mathcal{C}_n)_{n\in\mathbb{N}}$, and pick a representative $G'_n \subset G$ from each $\mathcal{C}_n$. Then $G_n := \bigcup_{j\leqslant n} G'_j$ has the desired properties. □

(We can achieve $G = \bigcup_{n\in\mathbb{N}} G_n$ if desired, by adding P_n to G'_n, but we will not need this.)

Given two graphs G, H of the same rank satisfying $|A(G)| = |A(H)|$ (for example G, H could be as in Lemma 5.1, or $G = H$), and a minor embedding $h : G < H$, note that h induces a bijection from $A(G)$ to $A(H)$ by Observation 2.9.

Definition 5.3. *We denote this bijection by* h^A*.*

Call a part P (as in Definition 4.3) of $H \in \mathrm{Rank}_\alpha$ *H-unstable*, if there is a sequence $(h_n)_{n\in\mathbb{N}}$ of minor embeddings $h_n : H < H$ such that each h_n maps P into a different part of H and h_n^A is the identity. Otherwise, we say that P is *H-stable*.

Lemma 5.4. *Let* H *be a rayless graph such that* $\mathcal{C}^\bullet(H)$ *is well-quasi-ordered. Then at most finitely many parts of* H *are* H*-stable.*

Proof. Suppose not, and let $(P_n)_{n\in\mathbb{N}}$ be an enumeration of the infinitely many H-stable parts of H. Since $\mathcal{C}^\bullet(H)$ is well-quasi-ordered, $(P_n)_{n\in\mathbb{N}}$ is good, and so by Observation 2.2, there is an infinite chain $P_{a_1} < P_{a_2} < \ldots$. Let $h_i : P_{a_i} < P_{a_{i+1}}$ be corresponding minor embeddings, and note that h_i induces a permutation π_i on $A(H)$ by Observation 2.10. By Lemma 4.8 we may assume, by passing to a subsequence if necessary, that each π_i is the identity on $A(H)$. Combining this with the Hilbert Hotel Principle as in the proof of Lemma 4.6 we can define $h : H < H$ such that $h(P_{a_i}) \subseteq P_{a_{i+1}}$ for every i, and h^A is the identity. Let $h_n, n \in \mathbb{N}$ be the composition of h with itself n times. Then $(h_n)_{n\in\mathbb{N}}$ witnesses that P_{a_1} is H-unstable, contradicting our assumption. □

Remark 5.0.3. Call a part P of $H \in \mathrm{Rank}_\alpha$ *redundant*, if $H < H \backslash P^c$, where $P^c := P \backslash A(H)$ denotes the co-part of P. Then Lemma 5.4 remains true if we replace 'H-stable' by 'irredundant'.

We can now prove the main result of this section.

Proof of Lemma 5.1. We will follow the approach of Lemma 4.6, and it is assumed that the reader is familiar with its proof. The main technical difficulty in comparison to that lemma will be handling the stable parts.

Again, the forward implication is trivial.

For the backward implication, let $(P_n)_{n \in \mathbb{N}}$ be an enumeration of the parts of G. Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of subgraphs of G as provided by Lemma 5.2. We may assume without loss of generality that $P_n \subseteq G_n$, because we can add $\bigcup_{i \leqslant n} P_n$ to G_n without increasing its rank.

Since $|A(G)| = |A(H)|$, every minor embedding $f : G_n < H$ induces a bijection f^A of $A(G)$ onto $A(H)$ as in Definition 5.3. Since $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(H)$, there is a sequence of minor embeddings $f_n : G_n < H$. We may assume without loss of generality that f_n^A is a constant bijection z, because we can achieve this by passing to a subsequence. We call any such sequence $(f_n)_{n \in \mathbb{N}}$ a *(G_n, z)-sequence*.

Easily, every co-part of G is mapped to a co-part of H by any marked minor embedding, and so we can adapt the definition of an h-stable part from Lemma 4.1 to any sequence $(g_n)_{n \in \mathbb{N}}$ of minor embeddings $g_n : G_n < H$: we call a part P of G *g-stable*, if $\bigcup_n g_n(P)$ is contained in the union of a finite set of parts of H, and call P *g-unstable* otherwise.

We claim that there is (G_n, z)-sequence $(g'_n)_{n \in \mathbb{N}}$ maximizing the set of unstable parts:

There is a (G_n, z)-sequence $(g'_n)_{n \in \mathbb{N}}$, $g'_n : G_n \to H$, such that every part P of G that is g^P-unstable with respect to some (G_n, z)-sequence $(g_n^P)_{n \in \mathbb{N}}$ is also g'-unstable. (11)

To see this, enumerate those parts P as $(P'_i)_{i \in \mathbb{N}}$, and form g'_n by picking infinitely many members from each $g^{P'_i}$, assuming without loss of generality that $g_n^{P'_i}(P'_i \backslash A(G))$ lie in distinct parts of H for different values of n. This proves (11).

Let $\mathcal{S}$ be the set of g'-stable parts of G, and $\mathcal{U}$ the set of all other parts of G. Let $\mathcal{S}'$ be the set of H-stable parts of H, and $\mathcal{U}'$ the set of all other parts of H. Let $H_S := \bigcup \mathcal{S}'$. By Lemma 5.4,

$$\mathcal{S}' \text{ is finite, and therefore } \mathrm{Rank}(H_S) < \mathrm{Rank}(H). \tag{12}$$

(This is why we need the assumption that $\mathcal{C}^\bullet(H)$ is well-quasi-ordered.)

Next, we claim that

$$g'_n(P^c) \subset \mathcal{S}' \text{ holds for almost all } n \text{ for every } P \in \mathcal{S}, \tag{13}$$

where $P^c := P \backslash A(G)$ is the co-part of P. To prove this, suppose to the contrary there is some $P \in \mathcal{S}$ such that $g'_n(P^c)$ is contained in an element U_n of $\mathcal{U}'$ for infinitely many values of n. Since P is g'-stable, $\bigcup_n g'_n(P^c)$ meets only finitely many parts of H, and so we may assume that these U_n coincide with a fixed

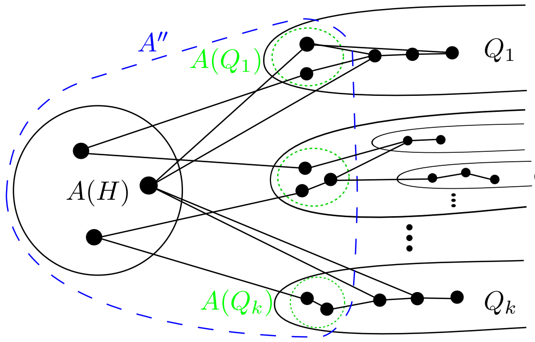

Figure 2: The vertex set $A'' := A(H) \cup \bigcup_{Q \in \mathcal{S}'} A(Q)$ in the proof of (15), enclosed by the dashed curve (blue).

$U \in U'$. Let $(h_n)_{n\in\mathbb{N}}, h_n : H < H$ be a sequence of minor embeddings witnessing that U is H-unstable, i.e. embedding U into infinitely many distinct parts of H, and such that h_n^A is the identity on $A(H)$. Then the sequence of compositions $h_n \circ g_n'$ embed P^c into infinitely many distinct parts of H, and so P is $(h \circ g')$-unstable. Here, we used the fact that $(h_n \circ g_n')^A = z$ by construction. But this contradicts our choice of g' since P is g'-stable and $(h \circ g')$ is a (G_n, z)-sequence. This contradiction proves (13).

Let $G_S := \bigcup \mathcal{S}$. We claim that

$$\text{Rank}(G_S) < \text{Rank}(H)(= \text{Rank}(G)). \tag{14}$$

For this, let $S_1, S_2, \ldots$ be an enumeration of $\mathcal{S}$, and let $G_n' := S_1 \cup \ldots \cup S_n$. We consider G_n' to be a marked graph, with $A(G)$ being the set of marked vertices. Using (g_n') it is not hard to obtain a sequence of marked-minor embeddings $g_n : G_n' < \bigcup \mathcal{S}'$. Indeed, for every n, and every $P \in \mathcal{S}$ such that $g_n'(P^c) \nsubseteq H_S$, we omit P^c from the domain of definition of g_n' to obtain a minor embedding g_n'' from a subgraph of G_n' to H_S. Note that each P is omitted for at most finitely many n by the previous statement. Then, we let (g_n) be a subsequence of (g_n''), chosen so that g_n does not omit any of $S_1, \ldots S_n$, which is possible since each S_i is omitted finitely often. By construction, (g_n) retains the property of (g_n') that each member induces the same bijection z of $A(G)$ onto $A(H)$, and g_n embeds G_n' into H_S.

We will now deduce (14) using (12) and the sequence (g_n). Let $\beta :=$ Rank(H_S). Let $A'' := A(H) \cup \bigcup_{Q \in \mathcal{S}'} A(Q)$ (Figure 2), and note that A'' is finite by (12), and that

$$\text{each component of } H_S - A'' \text{ has rank smaller than } \beta \tag{15}$$

by the definition of $A(Q)$.

Suppose Rank(G_S) equals Rank(G), which is larger than β by (12). Then there are infinitely many $P \in \mathcal{S}$ with Rank$(P^c) \geqslant \beta$ by the definition of rank. Let n be large enough that G_n' contains more than $|A''|$ such P's. Since the

P^c's are pairwise disjoint, by the pigeonhole principle, g_n maps at least one of them to $H_S - A''$, and in fact to a component C of $H_S - A''$ since each P^c is connected. This contradicts Observation 2.9, since $\mathrm{Rank}(P^c) \geqslant \beta > \mathrm{Rank}(C)$ by (15).

This contradiction proves (14). Thus $G_S \in \mathcal{C}(G)$, and so $G_S < G_n$ for some n, and in particular there is a minor embedding $g_S : G_S < H$, obtained by restricting some g_n. (We do not claim that $g_S(G_S)$ is contained in H_S.)

From now on we can proceed as in the proof of Lemma 4.6 to extend g_S to $g : G < H$ with $g^A = z$ by applying the Hilbert Hotel Principle to the g'-unstable parts of G. The only difference is that instead of appealing to the Graph Minor Theorem, we now use our assumption that $\mathcal{C}^\bullet(H)$ is well-quasi-ordered—combined with Observation 2.2— in order to find a chain $(Y_n)_{n \in \mathbb{N}}$ within any sequence of parts of H. $\square$

6 Countability of minor-twin-types of rayless forests

In this section we prove that for every ordinal $\alpha < \omega_1$ there are countably many minor-twin classes of countable forests of rank α (Theorem 6.3). This is an important step towards the proof of the backward direction of Theorem 1.4, which we will conclude in the next section. Our proof relies on Thomas' theorem that the class $TW(t)$ of countable graphs of tree-width at most t are well-quasi-ordered for every $t \in \mathbb{N}$ [18, (1.7)]. Although we are mainly interested in forests, our proof employs an intricate inductive argument for which it is essential to consider $TW(t)$ for every t. For a similar reason, we have to consider marked graphs even though our focus is on unmarked ones. Our final result will apply to $TW(t)$, not just the forests.

We prepare our proof with two lemmas. The first extends Thomas' aforementioned result to marked graphs.

Lemma 6.1. *For every $t \in \mathbb{N}$, the class of marked graphs (G, M) with $G \in TW(t)$ is well-quasi-ordered under $<_\bullet$.*

This is easily proved by replacing each marked vertex by a complete graph of the right size:

Proof. Let $((G_i, M_i))_{i \in \mathbb{N}}$ be a sequence of marked graphs in $TW(t)$. We need to show that it is good. Easily, we can assume that each G_i has more than $t+2$ vertices. Let $(G'_i, M'_i) := S^\bullet(S^\bullet((G_i, M_i)))$ —the marked suspension as defined in Section 2.2.1— and note that G'_i is 2-connected, and $TW(G'_i) \leqslant t+2$. Applying Lemma 2.6 twice, we deduce that $((G_i, M_i))_{i \in \mathbb{N}}$ is good if $((G'_i, M'_i))_{i \in \mathbb{N}}$ is, and so it remains to prove the latter.

Next, for each i, we modify (G'_i, M'_i) into an unmarked graph G''_i of tree-width $t+3$ by attaching a copy of K_{t+4} (which has tree-width $t+3$) to each $v \in M'_i$ by identifying v with an arbitrary vertex of K_{t+4}. We claim that for every $i, j \in \mathbb{N}$,

$$G''_i < G''_j \text{ if and only if } (G'_i, M'_i) <_\bullet (G'_j, M'_j). \tag{16}$$

This claim implies our statement, because $(G''_i)_{i \in \mathbb{N}}$ is good by Thomas' aforementioned theorem, implying that $((G'_i, M'_i))_{i \in \mathbb{N}}$ is good too.

The backward implication of (16) is trivial (and not needed for our proof). For the forward implication, suppose $\mathcal{B}$ is a minor model of G''_i in G''_j. Since G'_i is a block of G''_i, Lemma 2.5 says that $\mathcal{B}$ can be restricted into a minor model $\mathcal{B}'$ of G'_i within a block of G''_j. The latter block can only be G'_j, because $|G'_i| > t+4$ by our assumption, and all other blocks of G''_j are copies of K_{t+4}. Moreover, applying Lemma 2.5 to each other block B of G''_i, which is a K_{t+4}, we deduce that for each $v \in V(B)$, the branch set B_v contains a vertex of a single block B' of G''_j. This B' must be a K_{t+4} because $TW(B) = t+3 > TW(G'_j)$. Thus $\mathcal{B}$ induces a 1-1 correspondence between the $t+4$ vertices in B and the $t+4$ vertices in B'. It follows that the branch set of the unique vertex in $V(B) \cap M'_i$ contains the unique vertex in $V(B') \cap M''_j$. This means that $\mathcal{B}'$ maps each marked vertex of G'_i to a branch set containing a marked vertex of G'_j, which proves (16). □

For our proof of Theorem 1.4 in the next section we will need to prove that there are only countably many minor-twin types of countable forests of any given rank $\alpha < \omega_1$. The ideas involved are similar to the proof of Corollary 4.2, except that instead of $\mathcal{C}(G)$ we will be working with $\mathcal{C}^\bullet(G)$, and therefore with marked graph. Moreover, instead of the Graph Minor Theorem, we now need to use Nash-Williams' theorem [8] that the countable trees are well-quasi-ordered. We have made one step towards adapting our tools to marked graphs with Lemma 6.1, but we will need more: using the same unmarking idea as in the previous proof, we will next upper-bound the number of minor-twin classes of *marked* forests, and more generally graphs in $TW(t)$, by the number of minor-twin classes of *unmarked* graphs in $TW(t+1)$:

Lemma 6.2. *For every ordinal $\alpha < \omega_1$, and every $t \in \mathbb{N}_{>0}$, we have* $|\mathrm{Rank}^\bullet_\alpha \cap TW(t)|_{<_\bullet} \leqslant |\mathrm{Rank}_\alpha \cap TW(t+1)|_{<}$.

This lemma is the reason why it does not suffice to use Nash-Williams' theorem about forests, and instead we need Thomas' extension to $TW(t)$.

Proof. Pick a representative (G_X, M_X) from each marked-minor-twin class X of $\mathrm{Rank}^\bullet_\alpha \cap TW(t)$, and use it to define the unmarked graph $G'_X \in \mathrm{Rank}_\alpha \cap TW(t+1)$ similarly to the proof of Lemma 6.1, by attaching a copy of K_{t+2} to each marked vertex and unmarking it.

Suppose G_X, G_Y have the same number of marked vertices, say n. Similarly to (16), we claim that

$$G'_X < G'_Y \text{ if and only if } (G_X, M_X) <_\bullet (G_Y, M_Y). \tag{17}$$

From this we immediately deduce that G'_X, G'_Y are not minor-twins unless $X = Y$. This implies that each $n \in \mathbb{N}$ contributes at most $|\mathrm{Rank}_\alpha \cap TW(t+1)|_<$ to the count of classes in $\mathrm{Rank}^\bullet_\alpha \cap TW(t)$, and since the former is infinite, we obtain the desired inequality.

Thus it only remains to check (17). The proof is similar to that of (16), except that we now only apply Lemma 2.5 to the K_{t+2}'s of G'_X: if $\mathcal{B}$ is a minor model of G'_X in G'_Y, then it maps each copy of K_{t+2} in G'_X to one in G'_Y, whereby we use the fact that G_X has no K_{t+2} minor as its tree-width is less than that of K_{t+2}. It follows easily from this that for each $v \in M_X$ the branch set $\mathcal{B}(v)$ contains a vertex of M_Y. Moreover, if $w \in V(G_X) \backslash M_X$, then $\mathcal{B}(w)$ cannot intersect $G'_Y \backslash G_Y$, because all vertices in the latter subgraph are needed to accommodate the copies of K_{t+2} in G'_X. Thus by restricting $\mathcal{B}$ to G_X we obtain a marked-minor model of (G_X, M_X) in (G_Y, M_Y), proving (17). □

We are now ready to prove the main result of this section:

Theorem 6.3. *For every ordinal $\alpha < \omega_1$, and every $t \in \mathbb{N}_{>0}$, we have* $|\mathrm{Rank}^\bullet_\alpha \cap TW(t)|_{<_\bullet} = \aleph_0$.

Proof. We will prove the unmarked version

$$|\mathrm{Rank}_\alpha \cap TW(t)|_< = \aleph_0 \tag{18}$$

by a simultaneous (transfinite) induction on α and t. From this the statement follows immediately from Lemma 6.2.

For $\alpha = 0$, our claim (18) is trivially true for every t as there are countably many (marked) finite graphs; here we do not need the restriction on the tree-width.

For the inductive step $\alpha > 0$, we proceed by induction on t. Let Rank^n_α be the set of those $G \in \mathrm{Rank}_\alpha$ with $|A(G)| = n$. To prove that $|\mathrm{Rank}_\alpha \cap TW(t)|_<$ is countable, it suffices to prove that $|\mathrm{Rank}^n_\alpha \cap TW(t)|_<$ is countable for every $n \in \mathbb{N}$. Let $G \in \mathrm{Rank}^n_\alpha(TW(t))$, and apply Lemma 5.1 to deduce that the minor-twin class $[G]_{<_\bullet}$ is determined by $n = |A(G)|$ and the class $\mathcal{C}^\bullet(G)$; here, to be able to apply Lemma 5.1, we use the fact that $\mathcal{C}^\bullet(G) \subseteq \mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)$ is well-quasi-ordered by Lemma 6.1, and our inductive hypothesis that $|\mathrm{Rank}^\bullet_\beta \cap TW(t)|_{<_\bullet}$ is countable for every $\beta < \alpha$.

Since $\mathcal{C}^\bullet(G)$ is well-quasi-ordered, we can write $\mathcal{C}^\bullet(G)$ as $\mathrm{Forb}(X) \cap \mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)$ for some finite $X \subset \mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)$, whereby we used the fact that $TW(t)$ is minor-closed. We claim that $|\mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)|_{<_\bullet}$ is countable. Indeed, for every $\beta < \alpha$, Lemma 6.2 yields $|\mathrm{Rank}^\bullet_\beta \cap TW(t)|_{<_\bullet} \leqslant |\mathrm{Rank}_\beta \cap TW(t+1)|_<$, which is countable by our inductive hypothesis on α. Taking the union over all $\beta < \alpha$, which are countably many as $\alpha < \omega_1$, establishes our claim that $|\mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)|_{<_\bullet}$ is countable. Thus there are countably many ways to choose its finite subset X from above, hence countably many ways to choose $\mathcal{C}^\bullet(G)$, and therefore $[G]_<$. This proves (18). □

7 Proper minor-closed classes of rayless forests are Borel

In this section we use Theorem 6.3 to prove Theorem 1.4, which we restate for convenience:

Theorem 7.1. *Let $\mathcal{T} \subset \mathcal{G}$ be a minor-closed family of $\mathbb{N}$-labelled rayless forests. Then $\mathcal{T}$ is Borel if and only if it does not contain all rayless forests.*

The connection between minor-closed families and Borel sets is established by the following result that we will use to prove Theorem 7.1:

Theorem 7.2 ([4]). *Let G be a countable graph, and let $\mathcal{C} \subset \mathcal{G}$ denote the set of countable $\mathbb{N}$-labelled graphs that are isomorphic to a minor of G. Then $\mathcal{C}$ is a Borel subspace of $\mathcal{G}$.*

We will also need the following lemma, which is perhaps well-known when restricted to trees, but we include a proof for completeness.

Lemma 7.3. *For every countable rayless tree T there is an ordinal $\alpha(T) < \omega_1$ such that $T < G$ holds for every graph G with* $\text{Rank}(G) > \alpha(T)$.

Proof. We will state a modified statement that will help us apply transfinite induction on $\text{Rank}(T)$. A *rooted tree* (T, r) is a tree T with one of its vertices r designated as the root. The *tree-order* $\leqslant_r$ on $V(T)$ is the partial order defined by setting $x \leqslant_r y$ for any two vertices x, y such that x lies on the unique path in T from r to y. Given rooted trees $(T, r), (T', r')$, we write $(T, r) \preceq (T', r')$ if there is a subgraph embedding h of T into T' that respects the tree-order, i.e. $x \leqslant_r y$ implies $h(x) \leqslant_{r'} h(y)$ for every $x, y \in V(T)$. We will prove:

For every countable rayless rooted tree (T, r) there is an ordinal $\alpha = \alpha(T) < \omega_1$ such that $(T, r) \preceq (T_\alpha, r_\alpha)$ holds, (19)

where T_α denotes the minimal tree of Rank α, as provided by Observation 2.11, with rooted r_α provided in its construction.

For finite T it is not hard to see, by induction on the size of T, that $\alpha(T) = \omega$ suffices.

For $\text{Rank}(T) \geqslant 1$, the inductive hypothesis is easier to apply when $A(T) = \{r\}$, but this need not be the case. Therefore, we introduce the *spread* $S(T, r)$ of a rooted tree (T, r), defined as $S(T, r) := \max_{x \in A(T)} d(x, r)$, where d denotes the graph distance. Fixing $\text{Rank}(T)$, we prove (19) by induction on $S(T, r)$ as follows.

Let $C_1, C_2, \ldots$ be a (possibly finite) enumeration of the components of $T - r$, and root each C_i at the unique neighbour r_i of r in C_i. We claim that

for every i, either $\text{Rank}(C_i) < \text{Rank}(T)$, or $S(C_i, r_i) < S(T, r)$ (or both). (20)

To see this, suppose first that $C_i \cap A(T) = \varnothing$. Then $\text{Rank}(C_i) < \text{Rank}(T)$ because C_i is contained in a component of $T - A(T)$ in this case. Otherwise, let $A' := C_i \cap A(T) \neq \varnothing$. Then $\text{Rank}(C_i) = \text{Rank}(T)$, and it is not hard to check that $A(C_i) = A'$. Let $x \in A'$ be a vertex realising $S(C_i, r_i)$. Then $d(x, r) = 1 + d(x, r_i)$, implying the desired $S(T, r) > S(C_i, r_i)$.

Using (20), we can now define $\alpha := 1 + \sup_{i \in \mathbb{N}} \alpha(C_i)$, noting that $\alpha(C_i)$ is well-defined by induction on $S(T, r)$, nested inside our induction on $\text{Rank}(T)$. To start the induction on $S(T, r)$, we note that $S(T, r) = 0$ if and only if $A(T) = \{r\}$, in which case the first possibility always applies in (20), and therefore $\alpha(C_i)$ is well-defined by induction on $\text{Rank}(T)$.

We claim that $(T, r) \preceq (T_\alpha, r_\alpha)$. To prove this, we map r to r_α, and use our inductive hypothesis to embed each C_i into an appropriate component of $T_\alpha - r_\alpha$, rooted at the neighbour of r_α, preserving the tree order. The latter is possible because, by the construction of T_α, there are infinitely many components of $T_\alpha - r_\alpha$ isomorphic to $T_\alpha(C_i)$ for each i, and so we can pick a distinct such component to embed each C_i using our inductive hypotheses.

This proves (19). Our statement now follows by forgetting the root, and noting that any graph G with $\text{Rank}(G) > \alpha(T)$ has a component G' with $\text{Rank}(G') \geqslant \alpha(T)$ by the definition of rank, and G' contains T_α as a minor by Observation 2.11. □

We can now prove the main result of this section:

Proof of Theorem 7.1. If $\mathcal{T}$ is the family $\mathcal{F}$ of all rayless forests, then it is well-know that it is not Borel (in fact it is co-analytic complete) [6, 4].

So suppose $\mathcal{T}$ excludes some rayless forest T as a minor. Then $\mathcal{T}$ excludes a rayless tree, obtained from T by adding a vertex and joining it to each component, and so Lemma 7.3 implies that $\mathcal{T} \subseteq \mathrm{Rank}_\alpha \cap \mathcal{F}$ for some ordinal $\alpha < \omega_1$. Combining this with Theorem 6.3, we deduce that $\mathcal{T}$ consists of countably many minor-twin classes because there are countably many ordinals $\beta \leqslant \alpha$. Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be an enumeration of these classes, and pick a representative F_n from each $\mathcal{T}_n$. Let $\mathcal{C}_n \subset \mathcal{G}$ denote the set of countable $\mathbb{N}$-labelled graphs that are isomorphic to a minor of F_n. Easily,

$$\mathcal{T} = \bigcup \mathcal{C}_n, \tag{21}$$

because $\mathcal{T}$ is minor-closed, and $\bigcup \mathcal{C}_n$ contains a minor-twin of each element of $\mathcal{T}$. But each $\mathcal{C}_n$ is a Borel subset of $\mathcal{G}$ by Theorem 7.2, and therefore $\mathcal{T}$ is Borel by (21). □

Remark 7.0.1. If Thomas' conjecture, or its restriction to rayless graphs, is true, then following the lines of the proof of Theorem 7.1, but using Theorem 10.1 (a) $\leftrightarrow$ (i) instead of Theorem 6.3, we would deduce that if $\mathcal{T} \subset \mathcal{G}$ is a minor-closed family of $\mathbb{N}$-labelled rayless graphs that does not contain all rayless forests then $\mathcal{T}$ is Borel.

8 From marked to unmarked graphs

Marked graphs play an important role in the proof of Theorem 1.3, mainly via Lemma 5.1. This section provides two important tools for the former, which essentially allow us to ignore the marking in certain cases:

Lemma 8.1. *For every ordinal α, if $\mathrm{Rank}^\bullet_\alpha$ has a (infinite) descending chain, then so does Rank_α.*

Corollary 8.2. *For every $0 \leqslant \alpha < \omega_1$, we have $|\mathrm{Rank}_\alpha|_< = |\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$.*

Proof of Lemma 8.1. Suppose there is a $<_\bullet$-descending chain $G_1 \gneqq G_2 \gneqq G_3 \gneqq \ldots$ in $\mathrm{Rank}^\bullet_\alpha$, and let M_i denote the set of marked vertices of G_i. By Lemma 2.6 we may assume without loss of generality that each G_i is 2-connected, because we may add a couple of (marked) suspension vertices to each G_i without violating any of the relations $G_i \gneqq G_j$. Moreover, we may assume that $A(G_i)$ is 2-connected for every i, since every suspension vertex lies in $A(G_i)$. Here we use the obvious fact that $\mathrm{Rank}(S(G)) = \mathrm{Rank}(G)$ for every graph G.

Since $(G', M') <_\bullet (G, M)$ implies $|M| \geqslant |M'|$, we deduce that $|M_i|$ is monotone decreasing. Thus we may assume that it is constant (and at least 1, or there is nothing to prove). Similarly, since $(G', M') <_\bullet (G, M)$ implies $\mathrm{Rank}(G) \geqslant \mathrm{Rank}(G')$ (Observation 2.9), we may assume, by induction on α, that $\mathrm{Rank}(G_i) = \alpha$ for every i. Letting $A_i := A(G_i)$, it follows from Observation 2.10 that $|A_i|$ is monotone decreasing too, and again we may assume that it is constant. Similarly, we have $|A_i \cap M_i| \geqslant |A_{i+j} \cap M_{i+j}|$ for every $i, j \in \mathbb{N}$, and so we can also assume that $|A_i \cap M_i|$ is constant. Note that this implies that $|M_i \backslash A_i|$ is constant too, and that

any minor model $\mathcal{B}$ of G_i in G_{i-j} maps each vertex in $M_i \backslash A_i$ to a branch set intersecting $M_{i-j} \backslash A_{i-j}$. (22)

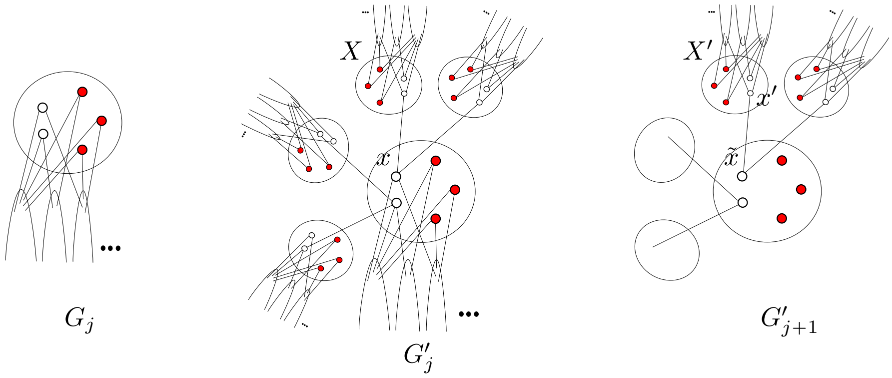

Figure 3: The graph G'_j in the proof of (24) (middle), produced by joining copies of G_j (left), and an attempt to embed it into G'_{j+1} (right).

We claim that

$$\text{we may assume that } M_i \subseteq A_i \text{ for every } i. \tag{23}$$

Indeed, if this is not the case, then let $\tilde{M}_i := M_i \backslash A_i$, and for each $x \in \tilde{M}_i$, attach a copy of the minimal tree T_α of Rank α, provided by Observation 2.11, to x, by identifying the root of T_α with x, to obtain the marked graph $(\tilde{G}_i, M_i \backslash \tilde{M}_i)$ for every $i \in \mathbb{N}$. We will show that $(\tilde{G}_n)_{n\in\mathbb{N}}$ is still a $<_\bullet$-descending chain: firstly, using (22) we can extend $\mathcal{B}$ into a minor model of $\tilde{G}_i$ in $\tilde{G}_{i-j}$. This shows that $\tilde{G}_{i-j} <_\bullet \tilde{G}_i$. To show that $\tilde{G}_i \nless_\bullet \tilde{G}_{i+j}$, suppose $\tilde{\mathcal{B}}$ is such a minor model. Then by Lemma 2.5, since $G_i \subset \tilde{G}_i$ is 2-connected, there is a model $\mathcal{B}'$ of G_i in a block of $\tilde{G}_{i+j}$. Since the only block of $\tilde{G}_{i+j}$ is G_{i+j} by construction, we deduce that $G_i <_\bullet G_{i+j}$, a contradiction that proves (23).

Next, we claim that

$$\text{we may assume that } M_i = A_i \text{ for every } i. \tag{24}$$

To prove this, we extend each G_i into a supergraph G'_i as follows. For each unmarked $x \in A_i$, add two disjoint copies of G_i to G_i, and join x to each of its two copies by an edge (Figure 3, middle). Having done so for each $x \in A_i \backslash M_i$ we have obtained G'_i. Note that each block of G'_i is isomorphic to G_i via an isomorphism that preserves the marking. To prove (24) it suffices to check that $G'_{j-1} \gneq G'_j$ still holds for every $j > 1$.

Let us first check $G'_j <_\bullet G'_{j-1}$. Pick a minor embedding $f : G_j < G_{j-1}$. Using (22) we can extend f into a minor embedding of G'_j in G'_{j-1} by embedding each copy of G_j attached to x to the copies of G'_{j-1} attached to the (unique) vertex $x' \in A_{j-1} \backslash M_{j-1}$ contained in $f(x)$, by imitating f inside these copies. This proves $G'_j <_\bullet G'_{j-1}$.

To check $G'_j \nless_\bullet G'_{j+1}$, suppose to the contrary there is a minor embedding $g : G'_j <_\bullet G'_{j+1}$. Recall that, by Observation 2.10, each $g(x), x \in A(G'_j)$ contains a distinct $x' \in A(G'_{j+1})$. Since $|A_i|$ and $|A_i \cap M_i|$ are constant, it follows that $|A(G'_i)|$ is constant too, hence $|A(G'_j)| = |A(G'_{j+1})|$. Thus there is a bijection $x \mapsto x'$ from $A(G'_j)$ to $A(G'_{j+1})$.

Recall we are assuming that A_j is 2-connected, and so g maps $A_j \subset A(G'_j)$ so that each branch set intersects a fixed block B of G'_{j+1} by Lemma 2.5. We claim that this block B must be G_{j+1} (rather than one of its copies in the construction of G'_{j+1}). Suppose to the contrary, there is $x \in A_j$ such that $x' \notin A_{j+1}$. Thus x' is a copy of an unmarked vertex of A_{j+1} (Figure 3, right). Note that the branch set $g(x)$ of x cannot contain the unique neighbour $\tilde{x}$ of x' in A_{j+1}, because $x \mapsto x'$ and so $\tilde{x}$ is in the branch set of some other vertex of $A(G'_j)$. Thus $g(x)$ avoids A_{j+1}. Since at most one of the two copies of G_j adjacent to x can contain the edge $x'\tilde{x}$, it follows that g maps at least one of these copies X inside the copy X' of G_{j+1} containing x'. But this copy avoids x' which is already used by $g(x)$, and we therefore reach a contradiction as we do not have enough vertices in $X' \cap A(G_{j+1})$ to accommodate $X \cap A(G_j)$.

Thus g maps A_j so that each branch set intersects G_{j+1}. Since $G_j \supset A_j$ is 2-connected, each branch set of a vertex of G_j intersects G_{j+1} by the first sentence of Lemma 2.5. By the second sentence, there is a model of G_j in G_{j+1} respecting the marking, a contradiction that proves (24).

Recall that, by Observation 2.10, any unmarked minor model $f : G_i < G_j$ is a marked minor model of $(G_i, A(G_i))$ in $(G_j, A(G_j))$. Thus (24) implies that $G_1 \gneqq G_2 \gneqq G_3 \gneqq \ldots$ is also an unmarked descending chain in Rank_α. $\square$

Remark 8.0.1. In this proof we only used the assumption that the family $(G_n)_{n\in\mathbb{N}}$ is a descending chain in order to make each of $|A(G_n)|, |M_n|$ and $\mathrm{Rank}(G_n)$ independent of n. Using this, we then produced a modified family $(G'_n)_{n\in\mathbb{N}}$ of unmarked graphs such that $G_i <_\bullet G_j$ if and only if $G'_i < G'_j$ for every i, j. This has Corollary 8.2 as an important consequence:

Proof of Corollary 8.2. Easily, $|\mathrm{Rank}_0|_< = |\mathrm{Rank}^\bullet_0|_{<_\bullet} = \aleph_0$ since there are countably many (marked) finite graphs, so assume $\alpha \geqslant 1$ from now on.

The inequality $|\mathrm{Rank}_\alpha|_< \leqslant |\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$ is trivial since each $<$-equivalence class of Rank_α is contained in a distinct $<_\bullet$-equivalence class of $\mathrm{Rank}^\bullet_\alpha$.

For the converse inequality, let $((G_i, M_i))_{i\in\mathcal{I}}$ be a family of marked graphs, one from each $<_\bullet$-equivalence class of $\mathrm{Rank}^\bullet_\alpha$. If $\mathcal{I}$ is countable then we are done since $|\mathrm{Rank}_\alpha|_<$ is infinite. If it is uncountable, then there is an uncountable subfamily $((G_i, M_i))_{i\in\mathcal{J}\subseteq\mathcal{I}}$ within which each of $|A(G_i)|, |M_i|$ and $\mathrm{Rank}(G_i)$ is constant, because there are countably many choices for each of these numbers. Using Remark 8.0.1 we can transform this subfamily into a family $(G'_i)_{i\in\mathcal{J}}$ of unmarked graphs of the same rank no two of which are minor twins. This completes the proof since $|\mathcal{J}| = |\mathcal{I}| = |\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$. $\square$

Remark 8.0.2. We cannot adapt Lemma 8.1 to antichains, because we cannot keep $|A_i(G)|$ constant.

9 Equivalences for a fixed rank

The following is the main technical ingredient of the proof of Theorem 1.3; it strengthens it by providing additional equivalent conditions, and refines it by layering graphs with respect to their rank. Let $\mathrm{Rank}^{n\bullet}_{<\alpha} := \{(G, M) \in \mathrm{Rank}^\bullet_{<\alpha} \mid |M| \leqslant n\}$. (We do not require $M \subseteq A(G)$ here.)

Theorem 9.1. *The following are equivalent for every ordinal* $1 \leqslant \alpha < \omega_1$*:*

(A) $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *is well-quasi-ordered for every* $n \in \mathbb{N}$*;*

(B) $|\mathrm{Rank}_\alpha|_<$ *is countable;*

(C) $|\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$ *is countable;*

(D) $|\mathrm{Rank}_\alpha|_< < 2^{\aleph_0}$*;*

(E) $\mathrm{Rank}^\bullet_\alpha$ *has no descending chain;*

(F) Rank_α *has no descending chain;*

(G) $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *has no antichain for every* $n \in \mathbb{N}$*;*

The proof of Theorem 9.1 is involved, and it is not easy to break its statement up into individual equivalences: not only we prove some of the equivalences by cycling through several items, but also to prove some of the implications we perform an induction on α that requires other implications.

Before proving Theorem 9.1, we introduce a way to represent a subclass of $\mathrm{Rank}^{n\bullet}_{<\alpha}$ by a single, unmarked, graph in Rank_α:

Definition 9.2. *Given a marked-minor-closed class* $\mathcal{C} \subseteq \mathrm{Rank}^{n\bullet}_{<\alpha}$*, we define a (unmarked) graph* $G_\mathcal{C} = G^n_\mathcal{C}$ *as follows.*

(i) for every marked-minor-twin class $\mathcal{H}$ *of elements of* $\mathcal{C}$*, we pick a representative* $(H, M) \in \mathcal{H}$*, and put* ω *pairwise disjoint copies of* H *into* $G_\mathcal{C}$*;*

(ii) we add a set $A_\mathcal{C}$ *of* n *isolated vertices to* $G_\mathcal{C}$*; and*

(iii) for each copy H_i *of* (H, M) *as in (i), and each of the (at most* n*) marked vertices* v *of* H_i*, we add an edge from* v *to a distinct element of* $A_\mathcal{C}$*.*

Note that $E(H_i, A_\mathcal{C})$ is a complete matching of the set $M(H_i)$ of marked vertices of H_i into $A_\mathcal{C}$. But $M(H_i)$ is empty for some H, and so $G_\mathcal{C}$ is disconnected. We perform step (iii) in such a way that

(iv) for each possible matching m *of* $M(H)$ *into* $A_\mathcal{C}$*, there are infinitely many indices* i *such that* $E(H_i, A_\mathcal{C})$ *coincides with* m*.*

Importantly, we have $\mathrm{Rank}(G_\mathcal{C}) \geqslant \mathrm{Rank}(\mathcal{C})$ and

$$A(G_\mathcal{C}) = A_\mathcal{C} \tag{25}$$

by construction. Moreover,

$$\mathrm{Rank}(G_\mathcal{C}) \leqslant \alpha, \tag{26}$$

because each component of $G_\mathcal{C} \backslash A_\mathcal{C}$ belongs to $\mathcal{C}$ and hence to $\mathrm{Rank}^\bullet_{<\alpha}$.

The following lemma shows that, under natural conditions, $G_\mathcal{C}$ is 'monotone' in $\mathcal{C}$.

Definition 9.3. *We say that* $\mathcal{C}$ *is* addable up to rank α*, if whenever* $(G_n)_{n\in\mathbb{N}}$ *is a sequence of graphs in* $\mathcal{C}$*, their disjoint union* $G := \dot{\bigcup}_n G_n$ *is also in* $\mathcal{C}$ *unless* $\mathrm{Rank}(G) \geqslant \alpha$*.*

For example, suppose $\mathcal{C} = \mathrm{Rank}_{<\alpha} \cap \mathrm{Forb}(H)$ for some connected graph $H \in \mathrm{Rank}_{<\alpha}$. Then it is easy to see that $\mathcal{C}$ is addable up to rank α, because $H < \dot{\bigcup}_n G_n$ only if $H < G_i$ for some i.

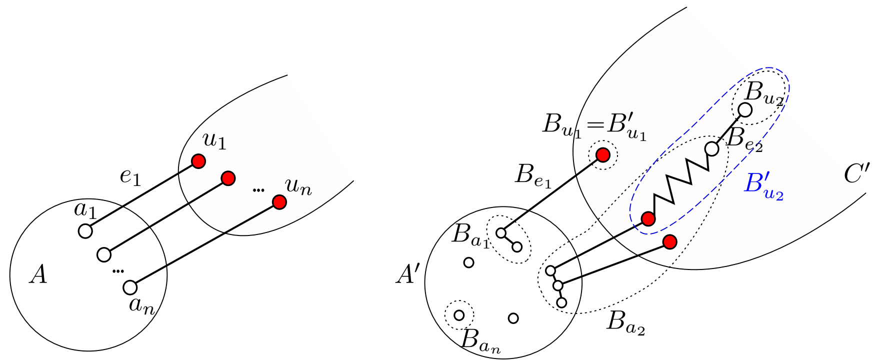

Figure 4: Defining B'_{u_i} in the proof of (27). The left picture depicts a portion of G, while the right picture depicts its minor model in G'. The dotted curves enclose the original branch sets, while the dashed curve (in blue) encloses B'_{u_2}.

Lemma 9.4. *Let $\mathcal{C} \subseteq \mathrm{Rank}^{n\bullet}_{<\alpha}$ and $\mathcal{C}' \subseteq \mathrm{Rank}^{m\bullet}_{<\alpha}$ be marked-minor-closed classes for some $n \leqslant m \in \mathbb{N}$, and suppose both $\mathcal{C}, \mathcal{C}'$ are addable up to rank α. Suppose* $\mathrm{Rank}(\mathcal{C}) = \mathrm{Rank}(\mathcal{C}') =: \beta \leqslant \alpha$. *Then $G_{\mathcal{C}} < G_{\mathcal{C}'}$ if and only if $\mathcal{C} \subseteq \mathcal{C}'$.*

We emphasize that the graphs $G_{\mathcal{C}}, G_{\mathcal{C}'}$ are unmarked, even though the classes $\mathcal{C}, \mathcal{C}'$ consist of marked graphs.

Proof. For the forward implication, suppose there is a minor model $\mathcal{B} = \{B_v \mid v \in V(G)\}$ of $G := G_{\mathcal{C}}$ in $G' := G_{\mathcal{C}}$. By (25) we have $A := A(G) = A_{\mathcal{C}}$ and $A' := A(G') = A_{\mathcal{C}'}$.

Pick $H \in \mathcal{C}$. We will prove $H \in \mathcal{C}'$, thus establishing the forward implication. We may assume that H is connected, because if each component of H lies in $\mathcal{C}'$ then so does H by the addability of $\mathcal{C}'$; indeed, if H has infinitely many components, then the supremum of their ranks is less than α since $H \in \mathcal{C}$. Let C be a component of $G \backslash A$ which is a marked-minor-twin of H, which exists by the construction of G and the connectedness of H. Recall that G contains infinitely many pairwise disjoint copies of C. Only finitely many of those can have a vertex the branch set of which intersects the finite set A', and so we may assume that $\mathcal{B}(C)$ avoids A'. Thus by Observation 2.4, $\mathcal{B}(C)$ —i.e. the submodel of $\mathcal{B}$ induced by C— is contained in a component C' of $G' \backslash A'$. To prove $H \in \mathcal{C}'$ it suffices to prove

$$C <_{\bullet} C', \tag{27}$$

since $H < C$ and $\mathcal{C}'$ is $<_{\bullet}$-closed. We will prove (27) by slightly modifying $\mathcal{B}$ and restricting it to C. This modification is needed to ensure that each marked vertex of C is mapped to a branch set containing a marked vertex of C'.

Let $\{u_1, \ldots, u_n\}$ denote the marked vertices of C, and recall that each u_i sends a unique edge e_i to A; let $a_i \in A$ denote the other end-vertex of e_i (Figure 4).

We consider the following two cases for every u_i:

If B_{e_i} has an end-vertex $a' \in A'$, then $B_{u_i} \cap B_{e_i}$ must be the unique neighbour of a' in C' because $\mathcal{B}(C)$ avoids A'. That neighbour is a marked vertex of C' by the construction of G'. In this case we let $B'_{u_i} := B_{u_i}$.

If not, then B_{e_i} lies in C'. In this case, let P_i be a path in G' from the vertex $B_{e_i} \cap B_{a_i}$ to A', which exists since B_{a_i} is connected and intersects A' by Observation 2.10. Note that the last edge of P_i joins a vertex $u'_i \in C'$ to a vertex of A', and therefore u'_i is a marked vertex of C'. Let $B'_{u_i} := B_{u_i} \cup B_{e_i} \cup (P_i \backslash A')$, and note that B'_{u_i} is connected and contains u'_i.

In both cases, B'_{u_i} contains a marked vertex. Easily, $B_{u_i} \cap B_{u_j} = \emptyset$ whenever $i \neq j$ as $P_i \subset B_{a_i}$ and $B_{a_i} \cap B_{a_j} = \emptyset$. Therefore, replacing each B_{u_i} by B'_{u_i}, and leaving $B'_x := B_x$ for every other vertex $x \in V(C)$, we obtain a minor model $\mathcal{B}'$ of C in C'. This proves (27).

For the backward implication, we assume $\mathcal{C} \subseteq \mathcal{C}'$, and construct an embedding of G into G' as follows. We begin by letting $B_a := a'$ for every $a \in A$, where $a \mapsto a'$ is an arbitrary but fixed bijection from A to a subset of A', which is possible since $|A| = n \leqslant m = |A'|$. For each 'part' H_i of G, find a part H' of G' such that $H_i <_\bullet H'$ and a model of H_i in H' such that the marked vertices of H_i are joined to A the same way as the marked vertices of H' are joined to A' with respect to $a \mapsto a'$. Since there are infinitely many such H' by item (iv) of Definition 9.2, we can choose them disjointly over the H_i's. □

We are now ready to prove the main result of this section.

Proof of Theorem 9.1. The equivalence of (B) and (C) follows from Corollary 8.2. The equivalence of (E) and (F) is Lemma 8.1. The implications (B) → (D) and (A) → (G) are trivial.

For the other implications, we apply induction on α. For $\alpha = 0$, we define $\text{Rank}_{<0}$ to be empty, and notice that all items are obviously true.

(A) → (B): The proof of this implication is very similar to that of Theorem 6.3.

To prove that $|\text{Rank}_\alpha|_<$ is countable, it suffices to prove that $|\text{Rank}^n_\alpha|_<$ is countable for every $n \in \mathbb{N}$, where $\text{Rank}^n_\alpha := \{G \in \text{Rank}_\alpha \mid |A(G)| = n\}$. Let $G \in \text{Rank}^n_\alpha$, and apply Lemma 5.1 to deduce that the minor-twin class $[G]_<$ is determined by $n = |A(G)|$ and the class $\mathcal{C}^\bullet(G)$. To be able to apply Lemma 5.1, we use the fact that $\mathcal{C}^\bullet(G) \subseteq \text{Rank}^{n\bullet}_{<\alpha}$ is well-quasi-ordered by our assumption, and that $|\text{Rank}^\bullet_\beta|_{<_\bullet}$ is countable for every $\beta < \alpha$ by our inductive hypothesis (A) → (C)—whereby we use the fact that $\text{Rank}^{n\bullet}_\beta$ is well-quasi-ordered since its superset $\text{Rank}^{n\bullet}_{<\alpha}$ is.

As $\mathcal{C}^\bullet(G)$ is well-quasi-ordered, we can express it as $\text{Forb}(X) \cap \text{Rank}^{n\bullet}_{<\alpha}$ for some finite $X \subset \text{Rank}^{n\bullet}_{<\alpha}$. Since $|\text{Rank}^{n\bullet}_\beta|_<$ is countable for every $\beta < \alpha$ as noted above, $|\text{Rank}^{n\bullet}_{<\alpha}|_<$ is countable too, being the sum of $|\text{Rank}^{n\bullet}_\beta|_<$ over the $\beta < \alpha$, which are countably many as $\alpha < \omega_1$. As $|\text{Rank}^{n\bullet}_{<\alpha}|_<$ is countable, there are countably many ways to choose its finite subset X from above, hence countably many ways to choose $\mathcal{C}^\bullet(G)$, and therefore $[G]_<$. This proves that $|\text{Rank}_\alpha|_<$ is countable.

(G) → (A): It suffices to show that $\text{Rank}^{n\bullet}_{<\alpha}$ has no $<_\bullet$-descending chain by Proposition 2.1, so suppose to the contrary $G_1 \gneqq G_2 \gneqq \dots$ is one. Then letting $\text{Rank}(G_1) =: \beta < \alpha$, and noting that $\text{Rank}(G_i) \leqslant \text{Rank}(G_1)$ by Observation 2.9

and $|A(G_i)| \leqslant |A(G_1)|$ by Observation 2.10, we deduce that this is a descending chain in $\mathrm{Rank}_\beta^{n\bullet}$. This contradicts the inductive hypothesis (G) $\to$ (E) for β.

(A) $\to$ (F): Suppose, for a contradiction, that Rank_α has a descending chain $G_1 \gneq G_2 \gneq G_3 \gneq \ldots$. By Observation 2.10 we may assume that $G_i \in \mathrm{Rank}_\alpha^n$ for a fixed $n \in \mathbb{N}$. Then, by Lemma 5.1, $\mathcal{C}^\bullet(G_1) \supsetneq \mathcal{C}^\bullet(G_2) \supsetneq \mathcal{C}^\bullet(G_3) \supsetneq \ldots$ is a descending chain of sub-classes of $\mathrm{Rank}_{<\alpha}^{n\bullet}$ with respect to containment. For each $k \in \mathbb{N}$, choose a marked graph $H_k \in \mathcal{C}^\bullet(G_k) \backslash \mathcal{C}^\bullet(G_{k+1})$, which is possible since $\mathcal{C}^\bullet(G_k) \supsetneq \mathcal{C}^\bullet(G_{k+1})$. Then (H_k) is a bad sequence in $\mathrm{Rank}_{<\alpha}^{n\bullet}$[4]. Indeed, if $H_k < H_{k+j}$ for some $k, j > 0$, then $H_k \in \mathcal{C}^\bullet(G_{k+j}) \subseteq \mathcal{C}^\bullet(G_{k+1})$ since $\mathcal{C}^\bullet(G_{k+j})$ is marked-minor closed. This contradicts that $\mathrm{Rank}_{<\alpha}^{n\bullet}$ is well-quasi-ordered.

The following two implications are similar to the forward direction of Corollary 4.2; the reader may want to recall that proof before reading them.

(F) $\to$ (A): Suppose, to the contrary, there is a bad sequence (H_i) in $\mathrm{Rank}_{<\alpha}^{n\bullet}$. We may assume without loss of generality that each H_i is connected by replacing it by $S^\bullet(H_i)$ and applying Lemma 2.6 (and increasing n by 1). Let $\mathcal{C}_i := \mathrm{Rank}_{<\alpha}^{n\bullet} \cap \mathrm{Forb}(H_1, \ldots, H_i)$ for each $i \in \mathbb{N}$. Note that $\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \ldots$ because $\mathcal{C}_i$ contains H_{i+1} but $\mathcal{C}_{i+1}$ does not. Clearly, $\mathcal{C}_i$ is closed under marked minors. Since the H_i are connected, each $\mathcal{C}_i$ is addable up to rank α as remarked after Definition 9.3. Let $G_i := G_{\mathcal{C}_i}$ be as in Definition 9.2. By our inductive hypothesis, (F) $\to$ (C) holds for every $\beta < \alpha$, and therefore $\mathcal{C}_i \subseteq \mathrm{Rank}_{<\alpha}^{n\bullet}$ consists of countably many minor-twin classes. Thus G_i is countable.

We claim that

$$\mathrm{Rank}(\mathcal{C}_i) = \alpha \text{ for every } i \in \mathbb{N}. \tag{28}$$

To see this, note first that if $\mathrm{Rank}(H_n) \leqslant \beta$ holds for some $\beta < \alpha$ and infinitely many $n \in \mathbb{N}$, then this subsequence of $(H_n)_{n\in\mathbb{N}}$ is a bad sequence in $\mathrm{Rank}_\beta^{n\bullet}$, contradicting our inductive hypothesis since Rank_β has no descending chain. Thus $\sup_n \mathrm{Rank}(H_n) = \alpha$. Since each $\mathcal{C}_i$ contains every $H_j, j > i$, we deduce that $\mathrm{Rank}(\mathcal{C}_i) \geqslant \alpha$. The converse inequality is obvious since $\mathcal{C}_i \subset \mathrm{Rank}_{<\alpha}^{n\bullet}$.

Lemma 9.4 now implies that $G_n \gneq G_{n+1}$ for every i, i.e. $(G_i)_{i\in\mathbb{N}}$ is an infinite descending $<$-chain in Rank_α, contradicting our assumption that none exists.

(D) $\to$ (G): Suppose $(H_n)_{n\in\mathbb{N}}$ is an anti-chain in $\mathrm{Rank}_{<\alpha}^{n\bullet}$. As above, we may assume that each H_n is connected by Lemma 2.6. Call a subset X of $\mathcal{H} := \{H_n \mid n \in \mathbb{N}\}$ *co-infinite*, if $\mathcal{H} \backslash X$ is infinite. Easily, there are $2^{\aleph_0}$ such X, because any subset of the even H_n's is co-infinite. We will follow the lines of the previous implication to produce $2^{\aleph_0}$-many graphs G_X none of which is a twin of another. Let $\mathcal{C}_X := \mathrm{Rank}_{<\alpha}^{n\bullet} \cap \mathrm{Forb}(X)$. Since each H_n is connected, $\mathcal{C}_X$ is addable up to rank α.

For each co-infinite $X \subset \mathcal{H}$, let $G_X := G_{\mathcal{C}_X}$ be as in Definition 9.2. Again, G_X is countable by our inductive hypothesis (D) $\to$ (C) and the fact that α is countable. Similarly to (28), we claim

$$\mathrm{Rank}(\mathcal{C}_X) = \alpha \text{ for every co-infinite } X \subseteq \mathcal{H}. \tag{29}$$

Indeed, $\mathrm{Rank}(\mathcal{C}_X) \leqslant \alpha$ is obvious, and to confirm $\mathrm{Rank}(\mathcal{C}_X) \geqslant \alpha$, we observe that each of the infinitely many graphs in $\mathcal{H} \backslash X$ is a minor of G_X by construction.

[4]This idea also appears in [2, LEMMA 6].

We claim that for every $\beta < \alpha$ there is a graph G in $\mathcal{H}\backslash X$ with $\text{Rank}(G) \geqslant \beta$. For if not, then $\mathcal{H}\backslash X$ is an anti-chain in $\text{Rank}^{n\bullet}_{<\beta}$ contradicting our inductive hypothesis. (This argument is the reason why we are working with co-infinite sets X.)

Thus by Lemma 9.4, G_X, G_Y are never minor-twins for co-infinite $X \neq Y \subset \mathcal{H}$, because $\mathcal{C}_X \neq \mathcal{C}_Y$ as $\mathcal{H}$ is an anti-chain. Thus we have obtained $2^{\aleph_0}$ distinct minor-twin classes, contradicting our assumption (D). □

We will use similar ideas to obtain two corollaries that we will also need for our main theorem in the next section:

Corollary 9.5. *Let $1 \leqslant \alpha < \omega_1$, and suppose*

$$\text{Rank}_\alpha \textit{ has no antichain of cardinality } 2^{\aleph_0}. \tag{30}$$

Then $\text{Rank}^{n\bullet}_{<\alpha}$ is well-quasi-ordered for every $n \in \mathbb{N}$.

Proof. We refine the proof of the implication (D) → (G) above to show that if $\text{Rank}^{n\bullet}_{<\alpha}$ has an anti-chain $(H_n)_{n\in\mathbb{N}}$ for some $n \in \mathbb{N}$, then (30) is contradicted. Indeed, given co-infinite sets $X, Y \subset \mathcal{H} := \{H_n \mid n \in \mathbb{N}\}$ note that $G_X < G_Y$ implies $\mathcal{C}_X \subseteq \mathcal{C}_Y$, which in turn implies $Y \subseteq X$. Thus if $\{X_i\}_{i\in\mathcal{I}}$ is a family of co-infinite sets that are pairwise $\subseteq$-incomparable, then $\{G_{X_i}\}_{i\in\mathcal{I}}$ is a $<$-antichain. Easily, there is such a family with $|\mathcal{I}| = 2^{\aleph_0}$, and so (30) implies that $\text{Rank}^{n\bullet}_{<\alpha}$ has no infinite anti-chain. By the equivalence of items (G), (A) of Theorem 9.1, (30) implies the stronger statement that $\text{Rank}^{n\bullet}_{<\alpha}$ is well-quasi-ordered for every $n \in \mathbb{N}$. □

Remark 9.0.1. It is not clear whether (30) can be added as a further equivalent condition in Theorem 9.1; we have just seen that it implies (A), but the latter only implies the weaker statement that $\text{Rank}_{<\alpha}$ has no infinite antichain.

Let $\mathcal{R}^\bullet$ denote the class of marked, countable, rayless graphs.

Corollary 9.6. *If $\text{Rank}^{n\bullet}_{<\gamma}$ is well-quasi-ordered for every ordinal $1 \leqslant \gamma < \omega_1$ and every $n \in \mathbb{N}$, then $\mathcal{R}^\bullet$ is well-quasi-ordered.*

Proof. The proof is similar to the implication (F) → (A) above, but we will have to increase the rank by one.

Suppose, to the contrary, there is a bad sequence $(H_n)_{n\in\mathbb{N}}$ in $\mathcal{R}^\bullet$. As ω_1 is a regular ordinal, and this sequence is countable, we deduce that there is $\alpha < \omega_1$ such that $H_n \in \text{Rank}^\bullet_{<\alpha}$ for every n. As usual, we may assume that each H_i is connected by replacing each H_i by $S^\bullet(H_i)$ and applying Lemma 2.6.

Let $\mathcal{C}_i := \text{Rank}^{n_i\bullet}_{<\alpha} \cap \text{Forb}(H_1, \ldots, H_i)$ for each $i \in \mathbb{N}$, where n_i is the maximum number of marked vertices in $\{H_1, \ldots, H_i\}$. We can no longer claim that $\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \ldots$, because graphs in $\mathcal{C}_j$ can contain more marked vertices than those in $\mathcal{C}_{j-1}$. To amend this, we introduce $\mathcal{C}^k_i := \mathcal{C}_i \cap \text{Rank}^{k\bullet}_{<\alpha}$ for every $k \in \mathbb{N}$. We now have $\mathcal{C}^k_1 \supseteq \mathcal{C}^k_2 \supseteq \ldots$, and $\mathcal{C}^{i+\ell}_i \supsetneq \mathcal{C}^{i+\ell}_{i+1}$ for every fixed i and large enough ℓ, because $\mathcal{C}^{i+\ell}_i$ contains H_{i+1} but $\mathcal{C}^{i+\ell}_{i+1}$ does not.

Let $G_{\mathcal{C}^k_i}$ be as in Definition 9.2. By Theorem 9.1 (A) → (C) $\mathcal{C}^k_i \subseteq \text{Rank}^\bullet_{<\alpha}$ consists of countably many minor-twin classes, and so $G_{\mathcal{C}^k_i}$ is countable. Similarly to (28), we will prove that $\text{Rank}(\mathcal{C}^k_i) = \alpha$ for every $i, k \in \mathbb{N}$. Indeed, if $\text{Rank}(H_n) \leqslant \beta$ holds for some $\beta < \alpha$ and infinitely many $n \in \mathbb{N}$, then this contradicts our inductive hypothesis. Thus $\sup_n \text{Rank}(H_n) = \alpha$. Since each $\mathcal{C}^k_i, i \in \mathbb{N}$

contains every $H_j, j > i$ as an unmarked graph, we deduce that $\mathrm{Rank}(\mathcal{C}_i^k) \geqslant \alpha$. The converse inequality is obvious since $\mathcal{C}_i^k \subset \mathrm{Rank}_{<\alpha}$.

Let $G_n := \dot{\bigcup}_{k\in\mathbb{N}} G_{\mathcal{C}_n^k}$ for every $n \in \mathbb{N}$, and note that $\mathrm{Rank}(G_n) \leqslant \alpha + 1$. We claim that $(G_n)_{n\in\mathbb{N}}$ is a $<$-descending chain. Easily, we have $G_n > G_{n+1}$ by applying the backward direction of Lemma 9.4 componentwise, since $\mathcal{C}_n^k \supseteq \mathcal{C}_{n+1}^k$. One the other hand, if $G_n < G_{n+1}$ holds for some n, then for every k there is k' such that $G_{\mathcal{C}_n^k} < G_{\mathcal{C}_{n+1}^{k'}}$. But there is k large enough that $H_{n+1} \in \mathcal{C}_n^k$, while $H_{n+1} \notin \mathcal{C}_{n+1}^{k'}$ for every k'. This contradicts the forward direction of Lemma 9.4. Thus $(G_n)_{n\in\mathbb{N}}$ is a $<$-descending chain in $\mathrm{Rank}_{\alpha+1}$, which contradicts Theorem 9.1 (A) $\rightarrow$ (F).

□

10 Concluding the proof of Theorem 1.3

We can now complete the proof of our main Theorem 1.3 in a more detailed version. Recall that $\mathcal{R}$ denotes the class of countable rayless graphs, and $\mathcal{R}^\bullet$ the marked countable rayless graphs. Let $\mathcal{R}^{n\bullet}$ denote the class of countable rayless graphs with at most n marked vertices.

Theorem 10.1. *The following statements are equivalent:*

(a) $\mathcal{R}$ *is well-quasi-ordered;*

(b) $\mathcal{R}^\bullet$ *is well-quasi-ordered;*

(c) $\mathcal{R}^{n\bullet}$ *is well-quasi-ordered for every* $n \in \mathbb{N}$*;*

(d) $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *is well-quasi-ordered for every* $n \in \mathbb{N}$ *and* $\alpha < \omega_1$*;*

(e) $\mathcal{R}$ *has no descending chain;*

(f) $\mathcal{R}^\bullet$ *has no descending chain;*

(g) $\mathcal{R}$ *has no infinite antichain;*

(h) $\mathcal{R}$ *has no antichain of cardinality* $2^{\aleph_0}$*;*

(i) $|\mathrm{Rank}_\alpha|_<$ *is countable for every* $\alpha < \omega_1$*;*

(j) $|\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$ *is countable for every* $\alpha < \omega_1$*;*

(k) $|\mathrm{Rank}_\alpha|_< < 2^{\aleph_0}$ *for every* $\alpha < \omega_1$*.*

Proof. We trivially have (b)$\rightarrow$ (c) $\rightarrow$ (d), and the implication (d) $\rightarrow$ (b) is Corollary 9.6. Thus items (b), (c), (d) are equivalent.

We trivially have (b)$\rightarrow$ (a) $\rightarrow$ (e). By the implication (F) $\rightarrow$ (A) of Theorem 9.1, (e) implies (d). Thus (a), (e) are also equivalent to the above items.

The equivalence between (e) and (f) follows easily from Lemma 8.1, by noting that in any descending sequence the ranks are decreasing by Observation 2.10.

We trivially have (a) $\rightarrow$ (g) $\rightarrow$ (h). Corollary 9.5 provides the implication (h) $\rightarrow$ (d), thus adding (g), (h) to be above equivalences.

The equivalences between items (i), (j), (k) and (d) follow by applying Theorem 9.1 to every $\alpha < \omega_1$. □

11 Open problems

Our main open problem, motivated by Thomas' conjecture and Theorem 1.1 is

Problem 11.1. *Are the rayless graphs (of arbitrary cardinality) well-quasi-ordered?*

Thomas [17] provides an example of a bad sequence of graphs of the cardinality of the continuum, which however contain plenty of rays.

A well-known problem of R. Bonnet [12] asks whether every well-quasi-ordered poset is a countable union of better-quasi-ordered posets. Our Theorem 1.3 comes close to corroborating this for the poset of countable rayless graphs. This motivates:

Problem 11.2. *Is the class of countable, rayless, graphs a countable union of better-quasi-ordered subclasses?*

I find The following special case of Proposition 11.1 particularly interesting:

Problem 11.3. *Are the countable, planar, rayless graphs well-quasi-ordered?*

I expect that a positive answer to this would imply a positive answer to the following problem, by following the lines of the proof of Theorem 1.1 (and Corollary 3.1):

Problem 11.4. *Are the finite planar graphs better-quasi-ordered?*

Recall that Thomas proved that $TW(k)$ is well-quasi-ordered for every $k \in \mathbb{N}$ [18, (1.7)]. Let $TW_{<\infty} = \bigcup_{k \in \mathbb{N}} TW(k)$ be the class of countable rayless graphs of finite (but not bounded) tree-width. To appreciate the difficulty of the last two problems, the reader may try the following:

Problem 11.5. *Is $TW_{<\infty}$ well-quasi-ordered?*

Our last problems are motivated by Theorem 1.4.

Problem 11.6. *Let $\mathcal{C} \subseteq \mathcal{G}$ be a family of $\mathbb{N}$-labelled rayless graphs which is closed under minor-twins. Is it true that $\mathcal{C}$ is well-quasi-ordered if and only if $\mathcal{C} \cap \mathrm{Rank}_\alpha$ is Borel for every $\alpha < \omega_1$?*

(The forward direction is true.)

Similarly, one can ask

Problem 11.7. *Let Q be a family of finite graphs. Is it true that Q is better-quasi-ordered if and only if $\mathrm{Rank}^1_\alpha(Q)$ is Borel for every $\alpha < \omega_1$?*

Recall that Robertson, Seymour, & Thomas [13] wrote that there is not much chance of proving Thomas' conjecture, and even our restriction to the rayless case seems out of reach. Theorem 10.1 provides new tools for attacking it, and perhaps there is now a chance of disproving it:

Problem 11.8. *Is there a family of rayless $\mathbb{N}$-labelled graphs which is closed under minors, has rank less than ω_1, and is not Borel?*

A positive answer, combined with Theorem 7.2 and the implication (a) $\rightarrow$ (i) of Theorem 10.1, would disprove Thomas' conjecture.

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