Preliminary Generalisations of Separator Theorems for Non-Convex Intersection Graphs

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Contents

1 Introduction

Graphs serve as fundamental structures in mathematics, as they represent relationships between any number of elements in a set. Consequently, they represent things like social network connections, hyperlink rankings in search engines, optimised routes on a GPS and much more [6]. More specifically to this project, certain classes of graphs can be represented as shapes in $\mathbb{R}^d$. This paper will focus on such representations of graphs, known as intersection graphs.

1.1 Introducing Intersection Graphs

A simplistic type of intersection graph is a circle packing of a graph (also known as disc packings of graphs, or coin graphs). Such representations are constructed by assigning each vertex of a graph a circle of some chosen size, asserting that if there is an edge between any two vertices in the graph, the corresponding two circles in the representation share a point (in other words, the two circles have disjoint interiors but non-empty intersection). Crucially in packings, no intersection can contain more than one point, which is not true in general for intersection graphs.

Most famously, Paul Koebe in 1937 [7] laid out the first proof of the circle packing theorem, also known as the 'Koebe Representation'. The theorem assigns a circle packing representation to any planar graph, which is a graph which can be oriented in a plane such that the edges intersect only at vertices and with their endpoints. An example of a circle packing can be seen in Figure 1, which depicts a planar graph on five vertices being mapped to a circle packing with the representation φ_2. Also pictured is an example of an intersection representation φ_1 of the same graph, where the interiors of the intersecting circles have non-empty intersections. In this case, the φ_1 is a 3-thin intersection representation, each point belonging to at most 3 circles. Such geometric characterisations of graphs are the basis from which intersection graphs have been developed.

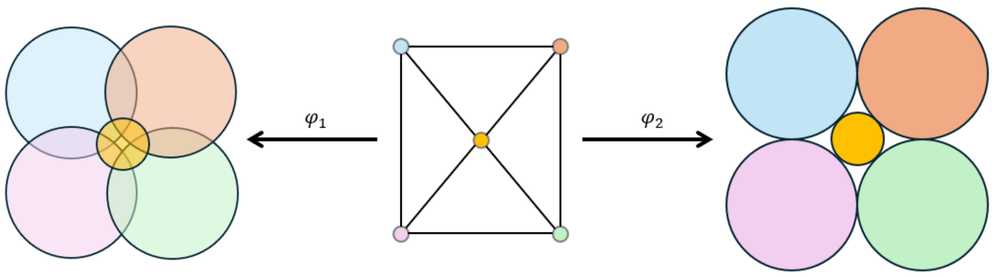

Figure 1: Example of a Circle Packing

1.2 Introducing Separator Theorems

Another key concept explored in this paper is the concept of separator theorems. A separator of a graph (sometimes called a vertex cut) is a subset of the vertices of a graph such that if one deletes all the vertices of this subset along with all of the vertices' adjacency relations, the new graph is a union of two or more disconnected components. Typically, people may impose a size restriction on these components; a commonly seen restriction is that the components have a size less than two-thirds of the number of vertices in the graph. This condition is what makes a separator *balanced*. Notably, a paper by Lipton and Tarjan (1979) [9] proved that all planar graphs have a separator of size $O(\sqrt{n})$ where O is the standard big-O notation depicting asymptotic growth rates. It may be relevant to note that this result was an improvement on a previous theorem by Ungar in 1951, who showed that a separator of size $O(\sqrt{n}\log(n))$ exists [10].

The most common and simplistic example of a vertex separation is the $n \times m$ grid graph depicting vertices in a grid of n rows and m columns, where an edge exists between two vertices if the vertices are adjacent to each other. An example of this can be seen in Figure 2, which depicts a 5×8 grid graph separated into 2 sets, A and B, both of which have size $\leq \frac{1}{2}V(G)$, separated by a set S, which has size 5. It should be somewhat intuitive that the size of the separator is $O(\sqrt{n})$ from Figure 2 since we can separate any $a \times b$ grid with $\min(a, b)$ vertices, where $V(G) = a \cdot b$

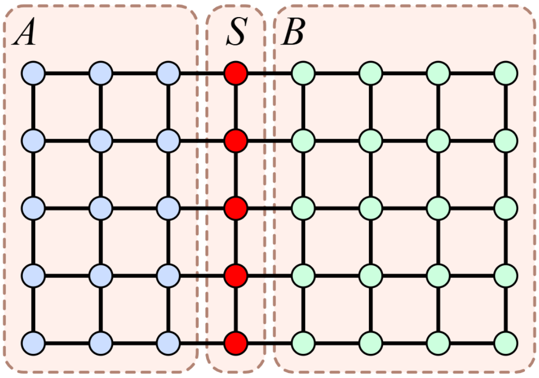

Figure 2: Grid Separator of a 5×8 Planar Grid Graph [4]

2 History and Applications

2.1 Historical Relevance of Separator Theorems

The Lipton and Tarjan algorithm for separating planar graphs was particularly useful in the aptly named 'divide and conquer' algorithms, wherein one 'divides' the problem into similar smaller problems, which are then combined to get a final solution. An example of such an algorithm is the merge sort algorithm, depicted in Figure 3, which divides a list up into single elements, and then merges sets pairwise in ascending order by comparing the smallest unsorted element in each set. The advantage of such an algorithm is that it scales a lot better than an algorithm which compares each element pairwise, as it requires exponentially fewer computations.

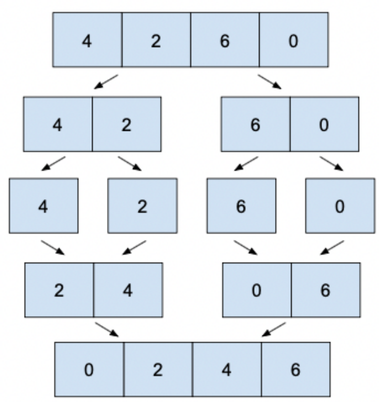

Figure 3: Example of a Merge Sort Algorithm

This can be seen in practice in Figure 3 where 2 comparisons are made in the first merge step, and 3 comparisons are made in the second one (5 in total), as opposed to making $\binom{4}{2} = 6$ comparisons.

In graph theory, this is done by recursively separating a graph to solve a graph-theoretic problem on much smaller graphs, which may be easier to do. Applications of such algorithmic problems can be as simple as in binary search algorithms and minimum spanning tree algorithms, or can be as complicated as being a structure for proofs in random graph theory and other involved areas of combinatorics. The result can also be used to approximate the solutions to NP-complete problems, Lower bounds of boolean circuit sizes, and maximum matching among many other applications listed with proofs in [8].

This result was extremely relevant not just in how it was used practically, but also in furthering the understanding of separator theorems and extending them to intersection representations. This paper will largely address how one can use intersection representations to prove that the graphs admitting such representations have separators of sublinear order. This result was proven for intersection representations using convex shapes in $\mathbb{R}^d$ by Dvořák, McCarty and Norin in 2021, in their paper, [3]. My paper attempts to extend the theory presented in [3] to certain types of non-convex sets.

2.2 A Discussion of the Primary Source

[3] states that non-convexity is ”somewhat troublesome” since one can represent arbitrarily large cliques using non-convex sets. For example, consider a long and skinny, L-shaped, non-convex set (See Figure 4). Then duplicate that shape and translate it to the right by the width of the top part of the L-shape, added to an arbitrarily small parameter $\epsilon_w > 0$. Doing the same with the height and $\epsilon_h > 0$ gives an intersection representation of the complete graph on 2 vertices (K_2). Repeating this process gives us an arbitrarily large clique which cannot be separated in a nice way: note that in order to create two disjoint sets in a clique, K_n, one would need to remove $V(K_n) - 2 = n - 2$ vertices.

Note that since the shapes being used are a choice, one can simply choose to make the L-shape skinnier than it is to accommodate more duplicates (as is done in Figure 4 for K_4), leading to the construction of arbitrarily large cliques in a *bounded* space. Another issue that [3] considers is the generation of $K_{n,n}$, which doesn't have a sufficiently nice way of being separated either. So if one can create arbitrarily large $K_{n,n}$ graphs as intersection representations, then we run into the same issue as before. One can create an arbitrarily large $K_{n,n}$ by considering 2 shapes that aren't superimposable. For example, shapes of the same size, rotated by an amount for which the shapes are not rotationally symmetric, as seen in Figure 5, can be used to construct representations of arbitrarily large complete bipartite graphs. Similar to the previous case, we can ensure that this representation is in a bounded space by making the L-shape 'skinnier' and reducing ϵ_h and ϵ_d by certain factors, as seen in Figure 5.

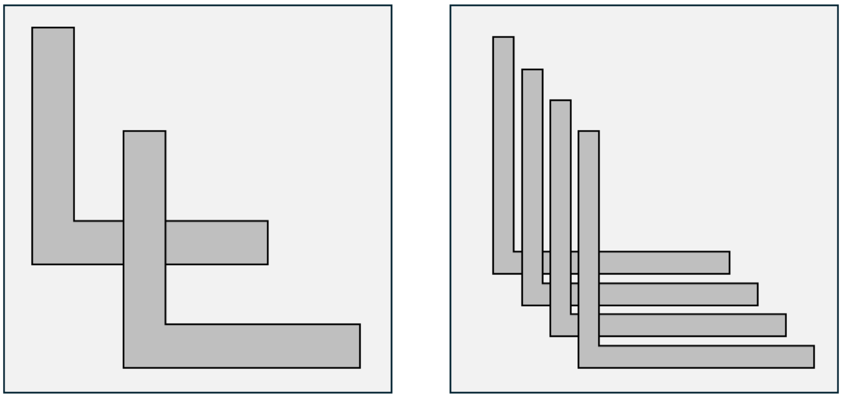

Figure 4: Intersection Representation of K_n with Non-Convex Shapes

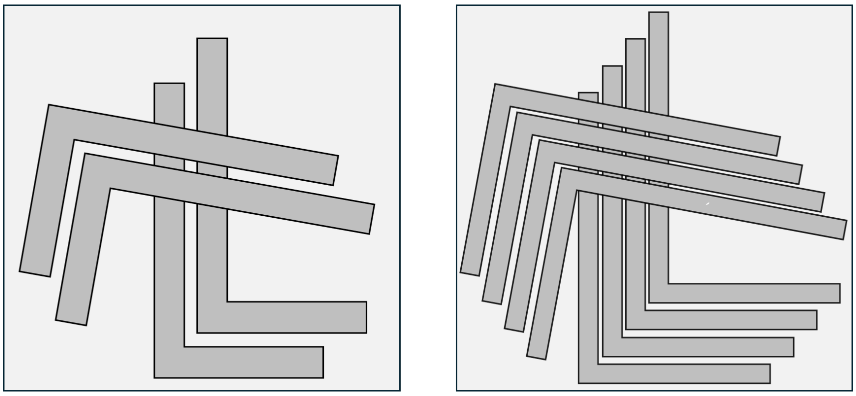

Figure 5: Intersection Representation of $K_{n,n}$ with Rotated Shapes

The first issue (Non-convexity) results in [3] discounting all non-convex sets in their paper. Notice that if one were to create a non-convex shape through the union of intersecting convex shapes, like balls in $\mathbb{R}^d$, then the proofs given in [3] can be generalised after making a few clever choices and considerations. As for the second issue, we will need to impose some kind of restriction on what kinds of shapes we can consider. Specifically for this paper, we will introduce a notion of comparability which will assert that the shapes are either oriented the same way, or have sizes that are different enough so that one shape contains the other, or some other sufficiently 'nice' relationship between any two shapes.

3 Formalisation and Set up

In this chapter, I will formalise all the notions that are relevant to the paper.

3.1 Formalisation of Relevant Objects and Sets

The first thing we will want to formalise is the definitions and properties of the shapes we consider in the construction of our intersection representations. We can start simple, formalising the notion of a shape.

Definition 3.1 (Non-Degenerate)**.** *A Euclidean space in $\mathbb{R}^d$ is said to be non-degenerate if it is not contained in a proper affine subspace [3].*

Definition 3.2 (Shapes)**.** *A shape is a Euclidean space that is closed, bounded and non-degenerate [3].*

Definition 3.3 ((Closed) Ball)**.** *A (closed) ball of radius r in $\mathbb{R}^d$ is a set of all points that are a distance at most r away from a centre x. Typically, we will set $x = \underline{0}$ without loss of generality*

Definition 3.4 (Ball-Like Shapes)**.** *For the purposes of this paper, we define a ball-like shape as a d-dimensional closed geometric shape, in which all planar sections of the shape are ellipses or circles. They are defined by d measurements, each representing the maximum displacement along a particular axis.*

Definition 3.5 (Centrally Symmetric)**.** *A shape is said to be centrally symmetric if it is invariant under reflection across the origin.*

Now we can start to discuss what kinds of shapes we are interested in for this paper.

Definition 3.6 (Convex Shapes)**.** *A shape S in $\mathbb{R}^d$ is called convex if for every x, y in the shape S, and $\lambda \in [0, 1]$, we have that $(1 - \lambda)x + \lambda y$ is in S. In other words, the line segment between x and y is contained within S.*

A shape is said to be *non-convex* if it does not satisfy this convexity property.

We can see that the L-shape described in Figure 4 is clearly not convex since we can find x and y in the shape such that the line from x to y, L_{xy}, is not contained in the shape. Note that we construct the L-shape from a union of 2 rectangles, and not with lines, to ensure that the space is non-degenerate. We also need to formalise the notion of a separator.

Definition 3.7 (Separators)**.** *For a graph, G, a set of vertices $X \subseteq V(G)$ is a f-separator that δ-splits G if the induced subgraph $G - X$ can be partitioned into two disjoint sets, A and B such that $|X| \leq f(n)$, and both $|A|$ and $|B| \leq \delta n$ for some function $f : \mathbb{N} \to \mathbb{N}$, and $0 < \delta < 1$.*

Note that as mentioned in Chapter 1, a separator is *balanced* if $|A|, |B| \leq \frac{2}{3}|V(G)|$. Also note that our definition accounts for *sublinearity* by specifying that $\delta < 1$, something that is explicitly described in [3].

3.2 Formalisation of Relevant Structures

We can now start defining the kinds of structures that will be investigated in this paper.

Definition 3.8 (Neighbourhood Systems)**.** *A c-thin neighbourhood system (also called a c-ply neighbourhood system) in d dimensions is a set of closed shapes, S, in $\mathbb{R}^d$ such that every point in $\mathbb{R}^d$ is in at most c shapes.*

Clearly a circle packing is a type of neighbourhood system where $c = 2$, $S = \{\text{Closed Balls in } \mathbb{R}^d\}$ and the shapes in S are arranged in a specific manner (such that the intersections of all the interiors are empty, and the intersections are solely in the boundary). Additionally, we will need to fully characterise the types of neighbourhood systems we want to use to represent the graphs we will eventually want to talk about. So far, we have a notion for the 'thin-ness' of a neighbourhood system, and a restriction on the kinds of shapes it uses, specifically, we assert that it must use shapes from a set of shapes, S. We still need to make sure that the shapes are sufficiently 'nice', which is done in Definition 3.16. We can then formalise the structures we talk about.

Remark 3.9. *The layout for these sections is chosen in this way to motivate the discussion of arbitrarily large complete graphs. Being unable to formalise the structure of the set S is a by product of this choice.*

Concepts that will be heavily discussed concern the amount of space any given shape takes up. From Chapter 1, it is clear that we will need to bound the overall size of the shapes we talk about as otherwise we can create arbitrarily large cliques as intersection representations. In fact, we will also need to bound the amount of 'non-convexity' a shape may have. In other words, we need to make sure that the non-convex shapes we concern ourselves with cannot become arbitrarily skinny to the point at which we can start making arbitrarily large cliques, as seen in Figure 4. This set-up requires and motivates the following definitions.

Definition 3.10 (Height)**.** *The height of a shape, B, is the diameter of the largest ball that is contained within B.*

This is an adjustment on the notion of height presented in [3], where height is described as the shortest distance between 2 hyperplanes that enclose the shape B. This definition is equivalent to the definition given above in the case of convex shapes, but doesn't work for non-convex shapes for the purposes with which this paper uses it. Therefore, this adjusted notion is required. A notion that we do borrow from [3] is the concept of the diameter of a shape, defined below.

Definition 3.11 (Diameter)**.** *The diameter of a shape, B, is the diameter of the largest ball containing B.*

Both these definitions are pictured in Figure 6 as h and d respectively.

Definition 3.12 (Aspect Ratio)**.** *The aspect ratio of a shape, B, is the ratio of the diameter of B, d_B, to the height of B, h_B. Mathematically, asp$(B) = \dfrac{d_B}{h_B} \geq 1$*

Conjecture 3.13. *Placing a finite real bound over the aspect ratio of shapes in S disallows the creation of arbitrarily large cliques.*

The above conjecture can be demonstrated by Figure 7, wherein we continue the discussion from

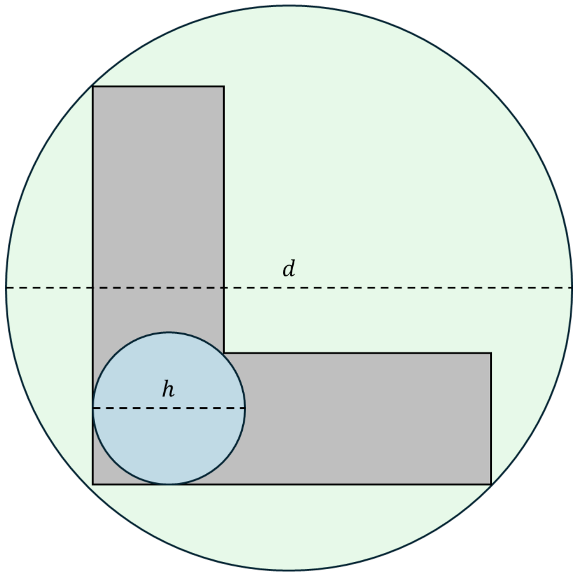

Figure 6: Picture Depicting the Height and Diameter

Chapter 1 on arbitrarily large cliques. By placing a bound on the aspect ratio of the L-shape, we limit how skinny either the top or the right side of the shape can be in relation to each other, since decreasing both arbitrarily would result in an arbitrarily small height and a diameter larger than the length of the shape, making the aspect ratio arbitrarily large. Given that the aspect ratio is bounded, the best we can do in way of a counterexample, is to make one of the ends skinny, and the other end as skinny as possible, so that it still contains a ball of diameter $d_{\text{asp}} = \dfrac{\text{asp}_{\text{bound}}}{\text{length}}$, which is the ratio between the bound we place on the aspect ratio, and the length of our shape. Note that if one end is skinny enough, the height is simply the thickness of the other end of the L-shape. In fact, this is the case if and only if the width of the skinny end is smaller than $\dfrac{d_{\text{asp}}}{2}$. Since in this case we are considering an arbitrarily small thickness, we can assume that it is less than $\dfrac{d_{\text{asp}}}{2}$.

This assertion now means that we can no longer create an arbitrarily large cliques K_n in a bounded space since there will be a maximum number of shapes containing a ball of diameter d_{asp} that we can fit in a bounded space to form a c-thin neighbourhood system, for d_{asp} and c finite, positive, non-zero real numbers.

3.3 Formalisation of Relationships Between Sets

Now we can start to address the issue of complete bipartite graphs. The final result we want to show is proven in Section 4 by Theorem 4.1, which is that when we consider shapes to represent vertices in a graph, we need to make sure that these shapes are sufficiently nice, so that they cannot make arbitrarily large complete bipartite graphs. However, we first need to define the kinds of properties we will need to impose on these shapes. The motivation for these properties follows from [3] and its notion of comparability. For the following definitions, let S be a set of shapes, and let B_1 and B_2 be shapes in this set. Also, let k and s be positive real numbers.

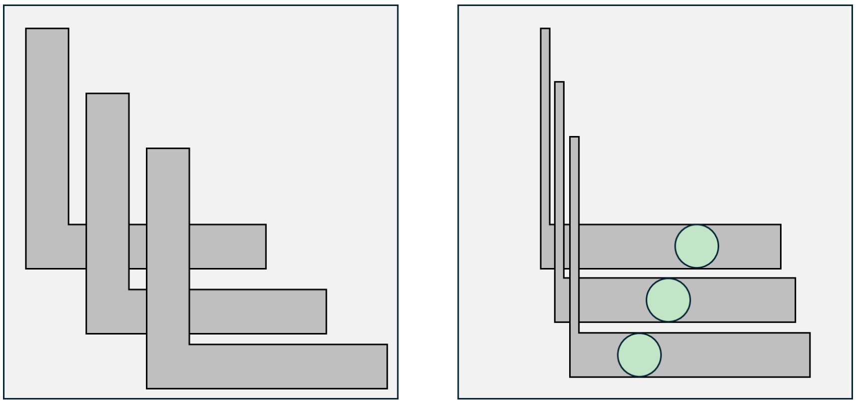

Figure 7: Picture Demonstrating the Effect of Bounding Aspect Ratio

Notation 3.14. *For a set B in $\mathbb{R}^d$, $kB = \{kx : x \in B\}$*

Definition 3.15 ($\leq_k$)**.** *For the two sets B_1 and B_2, we say $B_1 \leq_k B_2$ if there exists a translation of B_1 that is contained in kB_2.*

Definition 3.16 ($\sqsubseteq_s$)**.** *For the two sets B_1 and B_2, we say $B_1 \sqsubseteq_s B_2$ if for every point, $x \in B_2$, there exists a a translation of B_1, Call it B_1', such that x is in B_1' and the volume of the set $B_1' \cap B_2$ ($vol(B_1' \cap B_2)$) is at least the volume of B_1 divided by s $\left(\frac{1}{s} vol(B_1)\right)$*

Definition 3.17. *We call the two sets B_1 and B_2, $\leq_k$-comparable, if either $B_1 \leq_k B_2$ or $B_2 \leq_k B_1$.*

Definition 3.18. *Similarly, the two sets B_1 and B_2, are $\sqsubseteq_s$-comparable, if either $B_1 \sqsubseteq_s B_2$ or $B_2 \sqsubseteq_s B_1$.*

Examples of the 2 notions above can be seen for $d = 2$ in Figure 8 and Figure 9. In Figure 8, we can see examples of B_1 and B_2 on the left and for some $k \in \mathbb{R}^+$, a (green) kB_2 on the right. Specifically in this case, $k \approx 2.17$. On the right, we also see an (orange) translation of B_1 which fits inside kB_2, implying that in this case, $B_1 \leq_k B_2$.

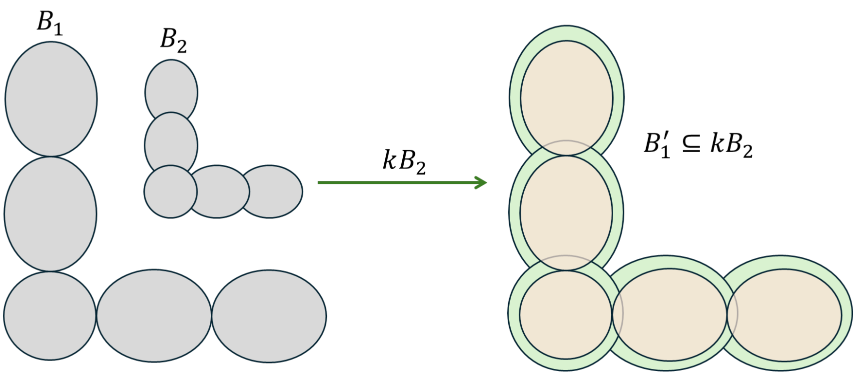

Figure 8: Example of $B_1 \leq_k B_2$

Similarly, in Figure 9, we can see examples of B_1 and B_2 on the left. On the right we see an example of a point chosen $x \in B_2$ to minimize the maximum value of vol$(B_1' \cap B_2)$ for any B_1' that contains this point, x. We do this to find the lowest possible value for s such that $B_1 \sqsubseteq_2 B_2$. After making such choices for x and B_1', we get the right side of Figure 9, and we can set $s > \dfrac{\text{vol}(B_1)}{\text{vol}(B_1' \cap B_2)}$. This results in $\text{vol}(B_1' \cap B_2) = \dfrac{\text{vol}(B_1' \cap B_2)}{\text{vol}(B_1)}\text{vol}(B_1) > \dfrac{1}{s}\text{vol}(B_1)$, which by definition implies that $B_1 \sqsubseteq_s B_2$.

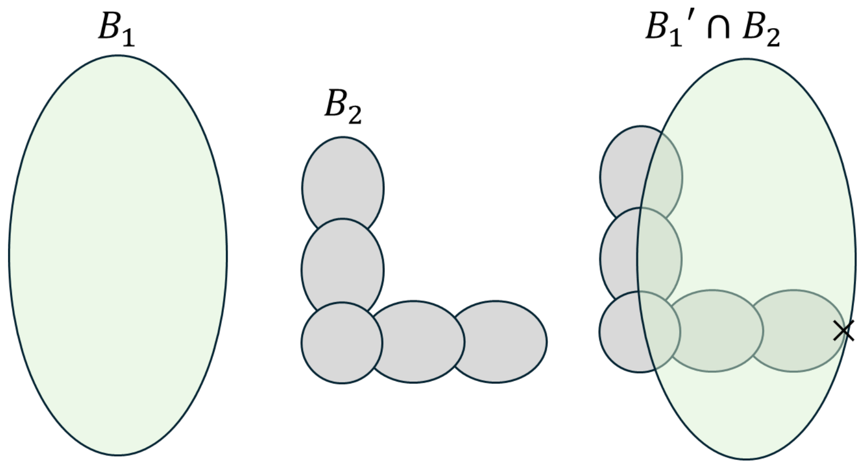

Figure 9: Example of $B_1 \sqsubseteq_s B_2$

These examples hopefully foreshadow the types of non-convex shapes we will be working with in this paper. We will need another such relation, given below, which will become relevant in Section 6

Definition 3.19 ($\leq_{k_1,k_2}$)**.** *For the two sets B_1 and B_2, we say $B_1 \leq_{k_1,k_2} B_2$ if for every point, $x \in k_1B_2$, there exists translations of B_1 and k_2B_1, call them B_1' and $(k_2B_1)'$ respectively, such that x is in $(k_2B_1)'$ and $B_1' \subseteq (k_2B_1)' \cap k_1B_2$*

It is perhaps relevant to see that Definition 3.16 and Definition 3.19 share some common elements in how they are defined. The link between these two definitions is further studied in Section 6. In particular, we show that these notions are equivalent up to a change in parameters.

We will now shift our discussion away from comparability and resume it in Section 4, and towards other relationships between sets that we will need for the goals we want to prove. We want to define the notion of generalised strong colouring numbers, which prompts the following definitions.

Definition 3.20 (Linear Orderings)**.** *A linear ordering, or a total ordering, of a set is an arrangement of elements in that set where any element either succeeds or precedes another element. Compared to a partial ordering, the distinguishing factor here is that all elements of the set are pairwise comparable.*

We will be placing a linear order on the vertices of a given graph in Section 5, so we will need some graph-theoretic properties as given below.

Definition 3.21. *A path, P_{uv}, from vertex u to v respects a linear order $\prec$, if for all vertices in the* *interior of the path, $w \in V(P_{uv})\backslash\{u, v\}$, v precedes w ($v \prec w$).*

Definition 3.22 (Strongly r-Reachable)**.** *A vertex u is strongly r-reachable from v with respect to $\prec$, if there is a path of length r from v to u that respects the linear ordering $\prec$*

Definition 3.23 (Generalised Strong r-Colouring Number)**.** *The definition of the generalised strong r-colouring number involves two sub-definitions:*

(i) $L_{G,\prec,r}(v)$ is the set of vertices that are r-reachable from v with respect to $\prec$

(ii) $col_{\prec,r}(G)$ is the maximum size of the set $L_{G,\prec,r}(v)$ over all the vertices in G

(iii) $col_r(G)$ is the minimum value of $col_{\prec,r}(G)$ over all linear orderings of $V(G)$

We call (iii) the generalised strong r-colouring number of a graph G.

Lastly, a lot of our proofs will utilise the concept that transforming a shape in ways that preserve certain properties. The transformations in question are defined below.

Definition 3.24 (Bijective Affine Transformations)**.** *Bijective affine transformations are types of bijective transformations which preserve parallelism and lines in Euclidean spaces.*

Definition 3.25 (Homothety)**.** *A homothety, or a homothetic transformation, is a transformation of an affine space which scales the space by a ratio with respect to a centre. Note that this transformation is what is being used in Notation 3.14, with the centre of the transformation set at the origin, and a scale k.*

3.4 Types of Shapes in the set S

We now formalise the notion of an intersection representation and begin a discussion on the types of shapes we might expect to see in our set S.

Definition 3.26 (Intersection Representation)**.** *An intersection representation of a graph G is a c-thin neighbourhood set of comprised from $n = V(G)$ shapes such that any two shapes have a non-empty intersection if and only if there exists an edge between the two vertices they represent.*

Now we adjust the terminology given in [3] to give the following definition.

Definition 3.27 ($(c, \sqsubseteq_s, S)$-tame Representations)**.** *We say that an intersection representation, $\varphi : V(G) \to \mathbb{R}^d$, is $(c, \sqsubseteq_s, S)$-tame representation of a graph G, if all of the following conditions are satisfied:*

(i) $\varphi(G)$ is a c-thin intersection representation

(ii) $\varphi(v)$ is a shape in S for all vertices v in the vertex set $V(G)$

(iii) $\varphi(v)$ and $\varphi(u)$ are $\sqsubseteq_s$-comparable for all vertices v, u in the vertex set $V(G)$

[3] proves that $(c, \sqsubseteq_s, S)$-tame representation has a balanced separator if S is the set of all convex shapes. In this paper, we attempt to show that this is the case for other sets which include shapes

that aren't convex.

However, generalising the notion of convexity is a considerably large undertaking and severely out of the scope of this essay. As a result, we will focus on furthering our understanding of the requirements of the set S by attempting to add certain shapes to it and try to examine how generally we can allow these shapes to be in our set, all the while ensuring that the theorems we rely on to prove (balanced) separability remain true.

Following in the direction we have already taken thus far, we want to add shapes like those seen in Figure 8. Such shapes are defined by a 'pivot' ball, and four appended ball-like shapes parallel to any 2 axes, depicted in Figure 10 for a two-dimensional example. For most of the coming sections, we will only be considering a union of five ball-like shapes, in the format described. It is worth noting, however, that concepts pertaining to such shapes will also apply to any other union of an odd number of balls with a pivot, unless explicitly stated otherwise. This idea is used in the corollaries following the main theorem, Theorem 4.1, of Section 4. We may also want to consider shapes with larger intersections. The theory we set up will focus largely on the 'pivot' ball, and so we can also consider shapes like B_2 in Figure 8, where each intersection contains more than a singular point. Intuitively, we are happy considering such shapes since we know that they are k-comparable with our standard union of ball-like shapes.

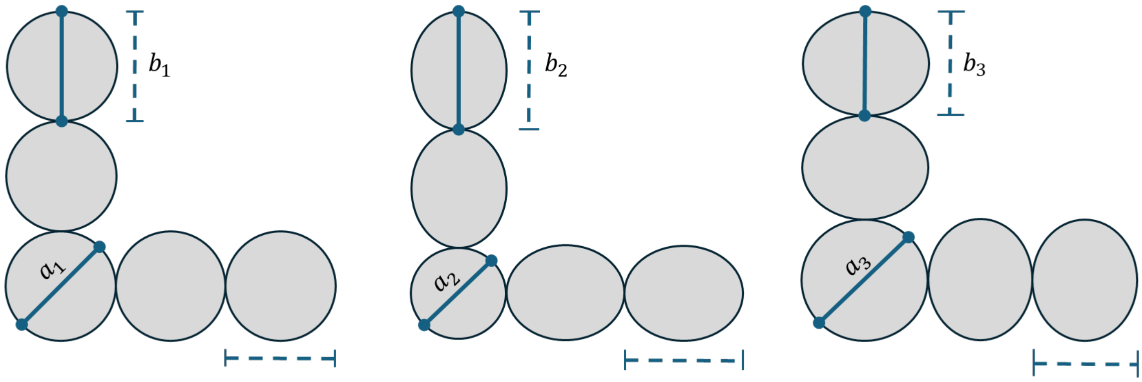

Figure 10: Examples of Shapes we could Consider to be in S

Note that there are likely a lot more shapes to be considered. Later sections introduce restrictions on the kinds of shapes we can consider, and while the above shapes are examples that we expect to adhere to these restrictions, there are almost definitely more shapes that have not been listed explicitly over here. In fact, even for convex shapes, [3] leaves the conditions required to ensure separability as an open question, having proven that $(c, \sqsubseteq_s, S)$-tame representations are a sufficient but not necessary condition for a graph to have a balanced separator.

Now we note some properties of the three given examples in Figure 10. First, the height of each shape is the diameter of the pivot ball, a. This is because the pivot ball is the ball of maximal diameter by construction, regardless of whether b is larger (like in the second example) or smaller (like in the third example) than a. In order to be completely rigorous, we need to provide d dimensions to define each ball-like shape here. Since we want the pivot ball to have a diameter a,

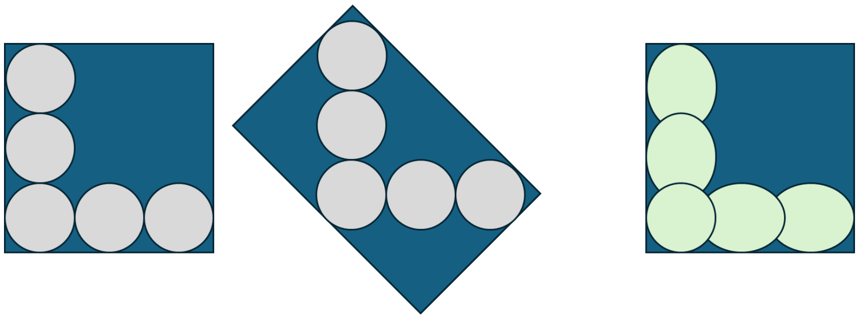

Figure 11: Picture depicting the difficulties in determining an envelope for shapes in S

we fix $d-1$ of the dimensions of the ball-like shapes as a, varying only one of their lengths (away from the pivot), b. Another thing that will be used in later sections is that the largest line segment contained within such shapes is of length $2b + a$. This means that in two dimensions, like in our examples, we can find a square of side length $2b + a$ that covers the shape completely. The ratio of this length to the diameter of the pivot ball, $\frac{2b+a}{a}$, also ends up being useful for the majority of Section 4.

The final idea we need to formalise before we can start to prove the necessity of $\leq_k$-comparability is the concept of an envelope. [3] gives the following definition:

Definition 3.28 (Envelope)**.** *An envelope of a shape B in $\mathbb{R}^d$ is a d-dimensional parallelepiped, T, such that $B \subseteq T$ and T has the smallest volume among parallelepipeds with this property.*

The tricky part for our set of shapes S is that we do not have a sufficiently 'nice' way to represent the volumetrically minimal parallelepiped that covers a shape. This problem is depicted in Figure 11, where in the left 2 sub-figures, it is unclear which rectangle is a better covering of the 2-dimensional shape. Although we can work this out computationally relatively simply, we cannot generalise this computation as easily for all the shapes in S. Notably, the right-most shape in Figure 11 would require a different algorithm to work out this minimal covering.

Even if we were able to compute these measurement sufficiently well, since we have forgone convexity in our hypotheses, we are no longer considering shapes that benefit from this minimality condition; in particular, these shapes cannot be applied to Lemma 8 or Lemma 9 in [3]. Instead, we will be interested in parallelepipeds that can be homothetically contracted to an envelope of a pivot ball, which is a convex shape, and use the property that the ball of largest diameter in a non-convex shape in B is the pivot ball, with a diameter of a. This motivates the following definition.

Definition 3.29 (Tight Box)**.** *A tight box of a shape B in $\mathbb{R}^d$ is the smallest hyperrectangle, T, containing the shape, $B \subseteq T$.*

As stated previously, since the ball-like shapes only vary in two dimensions, we can describe the dimensions and size of a tight box using just two variables, a and b. Going forward, we will only concern ourselves with these measurements in a plane for convenience, even though they will be relevant as measurements in $\mathbb{R}^d$

4 Necessity of $\leq_k$ Comparability

This chapter will attempt to clarify the types of non-convex shapes we would want to consider. Specifically, it will establish that shapes in S will need to be $\leq_k$ comparable. The implication being that if 2 shapes in S are not $\leq_k$ comparable, then we can create an intersection representation of a graph which does not have a (balanced) separator. For the rest of this section, when we refer to a shape in S, we are referring to a non-convex shape in S. For convex shapes, these relations are already proven in [3].

4.1 Supporting Lemmas

First, consider the set S is the set of all convex shapes and ball-like packings as seen in Figure 10. We will generalise this for all shapes containing a pivot ball, later in the Chapter. Accordingly, we will refer to a, b and p as per the discussion at the end of Section 3. Namely, that a is the diameter of the pivot ball of a shape in S, and b is the length of the ellipse in a planar section in the axis going away from the pivot ball. Accordingly $p = \frac{2b+a}{a}$. These notions are used throughout the following sections and will be referred to (in reference to a shape B in S) without explicit definition.

Notation 4.1. *This chapter and the following chapters will be addressing translates of multiple shapes as a consequence of addressing the necessity of $\leq_k$ Comparability of shapes. As a result, we use the notation K' to denote a translate of K.*

Lemma 4.2. *Let B be a non-convex shape in $S \subseteq \mathbb{R}^d$, and $p = \frac{2b+a}{a}$. If T is a tight box around the shape B, then $T \leq_{pd} B$*

Proof. This proof relies on the journal [1], where the following statement is proven:

> Let K be a convex body in E_n. Then K contains a parallelopiped P such that some translate of nP contains K

where E_n is being used to denote Euclidean space in n dimensions. Therefore, it is sufficient to show that shapes in S satisfy such a property after T is homothetically constricted. It is then perhaps trivial to notice that a homothetic dilation of the tight box $\tilde{T}$ around the circle of radius a, by a factor of $\frac{a+2b}{a} = p$ is equivalent to T. Furthermore, by Lemma 1 in [1], the circle of radius a is a convex shape in $\mathbb{R}^d$ and thus contains a parallelepiped K such that a translate of dK contains the circle. For this proof, we assume without loss of generality that this parallelepiped is the square $\tilde{T}$. And so, by a homothetic dilation, $\frac{1}{pd}T' = \tilde{T}' \subset B$ and $B' \subset pd\tilde{T} = T$. Thus $T \leq_{pd} B$ $\square$

In the case that $d = 2$, the proof follows exactly. It may be worth noting that in the case that $d > 2$, homothetically dilating the tight box around the pivot sphere, causes dimensions that aren't in the plane we concern ourselves with, to get scaled up to the interval $[\frac{-pda}{2}, \frac{pda}{2}]$ which is larger than it would need to be to contain the shape B (it would need to have an interval measurement larger than a, which holds since it has interval length of size $pda > a$ since $pd > 1$). Please note that such comments will be assumed throughout the paper, as we focus our energies towards proving more involved results.

As in the structure of [3], we will use our (adjusted) notion of height to set up the lemmas to come (*Recall the height of a shape is the diameter of the largest ball contained within it*). Accordingly, we define the following notation:

Notation 4.3. *For a shape $B \in \mathbb{R}^d$, we denote the height of B as h_B.*

Aside: In the journal [3], the authors concern themselves with envelopes of convex shapes, for example, B, which is different to our notion of tight boxes, particularly because they assert that envelopes are the volumetrically minimal parallelepipeds that cover the shape B. The notion of a tight box in this paper is not necessarily the parallelepiped of smallest volume that covers a shape $B^\star$ in S; it is probably not the case in most scenarios. What we would like to note, however, is that the lower bound for these 2 notions is the same, as that would prove to be the key factor in all of our proofs involving tight boxes.

Note that there are convex shapes with envelopes around them that have the same height as the shape itself, the simplest example of which is any ball. Also note that an envelope of a convex shape must have a height greater than or equal to the height of the convex shape itself, somewhat trivially. So the lower bound for the height of envelopes for convex shapes is just the height of the convex shape itself. On the other hand, for our shapes in S, one may be troubled to find out that certain shapes have envelopes of heights which are the same as the height of the shapes they cover, while the tight box covering such shapes have a larger height. An example of such a situation is given in Example 4.4, supported by the picture in Figure 12.

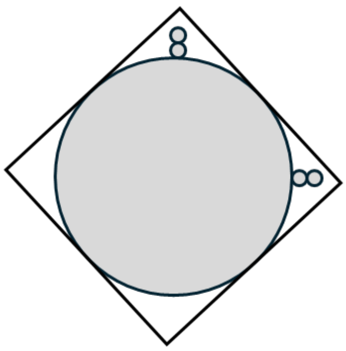

Figure 12: Picture Depicting Example 4.4

Example 4.4. *We define five balls in $\mathbb{R}^d$ with some intersection relationship between them. First, we take a unit circle (with unit diameter) centred at the origin. To this circle, we append 2 pairs of circles of diameter at most $\dfrac{\sqrt{d}-1}{4\sqrt{d}} = \sqrt{\dfrac{(\sqrt{d}-1)^2}{16d}}$ on any 2 axes (call these axes d_1 and d_2). Such a shape can be covered by a box of side length 1, centred at the origin and rotated 45° about the origin in the d_1-d_2 plane, pictured in Figure 12. Thus, such a shape, B, would have an envelope around it, T, wherein B has height $h_B = 1$ since the largest circle it contains has unit* *diameter, and T has height $h_T = 1$ trivially.*

In this example ($d = 2$), the tight box of the shape would have a height of $2\frac{\sqrt{d}-1}{4\sqrt{d}} + 1 = \frac{3\sqrt{d}-1}{2\sqrt{d}} \geq 1 = h_T$ where h_T is the height of the envelope of the shape. This demonstrates the differences between the two notions, which may appear problematic as the main source of this paper, [3], uses envelopes of convex shapes to prove most of their results, including the necessity of $\leq_k$ comparability.

However, this difference becomes inconsequential for us because we can consider appended balls that have arbitrarily small radii, which implies that for any $\epsilon > 0$, we can find a $\delta > 0$ such that if the radii of the appended balls are less than δ, then $h_T - h_B < \epsilon$. The specific computation of this $\delta(\epsilon)$ is quite trivial and so is omitted in favour of brevity. This continuity implies that we can consider limits as h_T approaches h_B, suggesting that the differences in the notions of tight boxes and envelopes are just a matter of accounting for the scaling, as done in Lemma 4.2.

Following this aside, we can prove more results about non-convex shapes in our set S.

Corollary 4.5. *For a non-convex shape $B \in S$ and its corresponding tight box, T, the height of the shape is a lower bound for the height of its tight box. Notationally, $h_B \leq h_T$.*

While trivial, following the discussion from the aside, it is relevant to note this relationship for the next lemma.

Lemma 4.6. *Every shape $B \in S \subseteq \mathbb{R}^d$, contains a ball of diameter $\frac{h}{pd}$ for $p = \frac{2b+a}{a}$ and some height $h = h_B$.*

Proof. Let T be the tight box around B. The height of T, h_T is at least $h = h_B$ as a result of Corollary 4.5. By Lemma 4.2, $T \leq_{pd} B$ and so $\frac{1}{pd}T'$ is contained in B. T' contains a ball of diameter at least h and so $\frac{1}{pd}T'$ contains a ball of diameter at least $\frac{h}{pd}$. □

4.2 Theorem and Corollaries

Theorem 4.1. *For our set S of shapes constructed by disk-packing a union of five ball-like convex shapes, m a positive integer, $p = \frac{2b+a}{a}$ and $k = 2p^3 d^{\frac{5}{2}} m$, if there exist at least 2 $\leq_k$-incomparable shapes $B_1, B_2 \in S$, then the complete bipartite graph, $K_{m,m}$, can be represented by a 2-thin intersection graph of shapes in S*

Proof. Assume for contradiction that the complete bipartite graph $K_{m,m}$ cannot be represented as an intersection graph of shapes from S. Consider the 2 shapes, B_1 and B_2 that are $\leq_k$ incomparable. This implies that neither $B_1 \leq_k B_2$ nor $B_2 \leq_k B_1$.

We set up the proof by considering a bijective affine transformation, $B_1^\star$ of B_1, which would place it inside the d-dimensional cube of side length p centred at the origin, $T_1^\star$. This cube is equivalent to the tight box around the transformed $B_1^\star$. From this point forward, we assume that the same bijective affine transformation is applied to the whole space, and accordingly, we will drop the $\star$-notation. We can do this since $B_1 \not\leq_k B_2$, implies that we have $B_1^\star \not\leq_k B_2^\star$ trivially by applying the same bijective affine transformations, and since these transformations are bijective, they have an inverse relation which we can apply to get the reverse implication (suggesting equivalence between the 2 representations).

Consider the ball U of diameter $p\sqrt{d}$. In this set up, $B_1 \subseteq T_1 \subseteq U$ and so $U \not\leq_k B_2$, since no B_1 translate is in B_2, and $B_1 \subseteq U$, so no U translate is in B_2. Therefore B_2 does not contain a ball of diameter $\frac{p\sqrt{d}}{k}$. This implies that B_2 has a height of less that $\frac{p\sqrt{d}}{k}$. By noting that $\frac{p\sqrt{d}}{k} \leq \frac{pd^{\frac{3}{2}}}{k}$, we conclude that there exists some vector v and a hyperplane H such that $|v| = \frac{pd^{\frac{3}{2}}}{k}$ and B_2 is contained within H and $H + v$.

Now we want to undo the translations in the affine maps, so we represent B_1 as $B_1' + q$ for some translation vector q. Since $T_1 \leq_{pd} B_1'$ for any translate of B_1, we get $q + T_1 \leq_{pd} q + B_1' = B_1$. So for some scaled translation vector $q^\diamond$, we have that $q^\diamond + \frac{1}{pd}T_1 \subseteq B_1$. So the ball $U' = q^\diamond + \frac{1}{p^2 d^{\frac{3}{2}}}U$ has diameter $\frac{1}{pd}$, and is contained in B_1 by Lemma 4.6, noting that in our affine map, B_1 has height, $h_{B_1} = 1$

We can now work with B_2. First we notice that $B_2 \not\leq_k U'$ since $U' \subseteq B_1$ and $B_2 \not\leq_k B_1$ so no translate of B_2 is contained in B_1 which implies that no translate of B_2 is contained in $U' \subseteq B_1$. Let s be the longest line segment in B_2 and let r be the length of s. Note that by choice of the longest segment, we choose an endpoint of s to be the centre of a circle with radius r that contains B_2. But $B_2 \not\leq_k U'$ and so no translate of B_2 is contained in a circle of diameter $\frac{k}{pd}$ which implies that $2r > \frac{k}{pd}$, or that $r > \frac{k}{2pd}$. By substituting the value for k in the statement of the theorem, we get the following:

$$r > \frac{2p^3 d^{\frac{5}{2}} m}{2pd} = p^2 d^{\frac{3}{2}} m > mp\sqrt{d}$$

This implies that there are m points in s that are separated by a distance of at least $p\sqrt{d}$. Call these points $\{z_1, z_2, ..., z_m\}$

Finally, let $L_i = B_1 - q^\diamond + z_i$ and $R_i = B_2 + (i-1)v$. It is immediately noticeable that $\{R_1, ..., R_m\}$ are all disjoint due to the fact that each R_i has an (outer) diameter which is smaller than the length of v, since the (outer) diameter of R_i is equal to the (outer) diameter of B_2, and each R_i is at least a distance of $|v|$ apart from each other. Furthermore, the set $\{L_1, ..., L_m\}$ contains elements that are all pairwise disjoint as well. This is because for all i in $\{1, ..., m\}$, we have that $L_i \subseteq U - q^\diamond + z_i$ because $B_1 \subseteq U$, and since U has a diameter less than $p\sqrt{d}$, and each z_i, z_j are at least a distance of $p\sqrt{d}$ away from each other by choice.

Now all that is left to show is that the intersection of any L_i and R_j is non-empty for any choice of $i, j \in \{1, ..., m\}$. Indeed, the point $z_i + (j-1)v$ is contained in both L_i and R_j. It is contained in R_i as a simple consequence of the fact that z_i is in B_2, and so $z_i + (j-1)v$ is in $B_2 + (j-1)v$.

$z_i + (j-1)v$ is contained in L_i with the following logic:

$$L_i = B_1 - q^\diamond + z_i \supseteq U' - q^\diamond + z_i = z_i + d^{\frac{-3}{2}} U'$$

so L_i contains a circle centred at z_i with diameter $\frac{1}{pd}$. This means that it has a height of at least $\frac{1}{pd}$, and covers (at least) a radius of $\frac{1}{2pd}$ around z_i. All the while, $z_i + (j-1)v$ represents a point that is a distance of $d(z_i, z_i + (j-1)v)$ away from z_i. Now note that:

$$d(z_i, z_i + (j-1)v) = d(0, (j-1)v) = ||(j-1)v||_2 \leq ||(m-1)v||_2 \leq m||v||_2$$

$$m||v||_2 = \frac{p^2 d^{\frac{3}{2}}}{k} = \frac{1}{2pd}$$

Therefore, $z_i + (j-1)v$ represents a point that lies at a distance less than $\frac{1}{2pd}$ away from z_i and so it lies within the circle of radius $\frac{1}{2pd}$ centred at z_i, which in turn lies in L_i. Thus, in a bijectively affine transformed space, we have found a 2-thin representation of $K_{m,m}$. By applying the inverse transformation, one can obtain the result in the original space. Therefore, what has been proven here is necessary and sufficient for the statement to be held. □

Some corollaries can come from this theorem, and they are listed as follows:

Corollary 4.7. *For the same set S of shapes constructed by a union of 5 balls, m a positive integer and $k = 54d^{\frac{5}{2}}m$, if there exist at least 2 $\leq_k$-incomparable shapes $B_1, B_2 \in S$, then the complete bipartite graph, $K_{m,m}$, can be represented by a 2-thin intersection graph of shapes in S*

Proof. In this corollary, we no longer consider ball-like shapes. As a result, we no longer have to deal with the ratio of the height of a tight box around shapes in S to the height of those shapes, as a variable of a and b. Instead, we know that for a ball $a = b$ and so $p = \frac{a+2b}{a} = \frac{3a}{a} = 3$. The result should now be trivial, following from Theorem 4.1 and the fact $p = 3$ which implies that $k = 2 \cdot 3^3 d^{\frac{5}{2}} m = 54d^{\frac{5}{2}} m$ □

Corollary 4.8. *For a set S of shapes constructed by disk-packing a union of an odd number, n, of balls in the same way as before, m a positive integer and $k = 2\left(\frac{n+1}{2}\right)^3 d^{\frac{5}{2}} m$, if there exist at least 2 $\leq_k$-incomparable shapes $B_1, B_2 \in S$, then the complete bipartite graph, $K_{m,m}$, can be represented by a 2-thin intersection graph of shapes in S*

Proof. Note that we constructed p from the ratio of the height of the tight box h_T to the height of its corresponding shape h_B. In other words, $p = \frac{h_T}{h_B}$ which was equal to $\frac{2b+a}{a}$ in the case of 5 balls. One can easily see that h_B is still equal to a, and h_T is still the sum of the lengths along the parallel axes. Thus $h_B = \frac{n-1}{2} b + a$. Since we consider balls in this corollary, we get that $h_B = \frac{n+1}{2} a$ by setting $b = a$. Thus $p = \frac{n+1}{2}$. Now using Theorem 4.1 with this new value of p, the proof follows identically. □

We can now move away from our restriction of disk-packings and consider 2-thin neighbourhood sets, as in shapes like B_2 in Figure 8. For clarity, we will call this set S'.

Corollary 4.9. *For a set S' of 2-thin neighbourhood sets constructed from 5 ball-like shapes, as in B_2 in Figure 8, m a positive integer, $p = \frac{2b+a}{a}$ and $k = 2p^3 d^{\frac{5}{2}} m$, if there exist at least 2 $\leq_k$-incomparable shapes $B_1, B_2 \in S'$, then the complete bipartite graph, $K_{m,m}$, can be represented by a 2-thin intersection graph of shapes in S'*

Proof. The first thing to notice here is that the statement in this corollary is almost identical to the one in Theorem 4.1. This is because in Lemma 4.2 and Lemma 4.6, we use $\leq_{pd}$ and $\frac{h}{pd}$ respectively in the statements. Naturally, if these lemmas hold true for S', the set of 2-thin neighbourhood sets as in the statement of this corollary, then the rest of the proof of Theorem 4.1 follows. So we are left with proving that a tight box, T, around $B \in S$ satisfies $T \leq_{pd} B$, and that every shape B contains a ball of diameter $\frac{h}{pd}$. These are easily proven after noticing that there is a mapping from S' to S by simply translating the appended balls upwards or to the right. For $B \in S'$, call the associated shape in S, C. Let the tight box around B and C be denoted as T_B and T_C, respectively. Note that $T_B \subseteq T_C$ by construction. Let C_p be the pivot ball of C. Then from the proof of Lemma 4.2, we can trivially see that $T_C \leq_{pd} C_p$. By noting that C_p is a subset of B, we can conclude that $T_B \subseteq T_C \subseteq pdC_p \subseteq pdB$ which implies that $T_B \subseteq pdB$, and specifically, $T_B \leq_{pd} B$. Using this and Lemma 4.6, the second part follows trivially since $h_B = h_C = a$. $\square$

Remark 4.10. *One perhaps important thing to note about this section is that in Lemma 4.2 and Lemma 4.6, we made the conditions weaker (by considering $\leq_{pd}$ and $\frac{h}{pd}$), and so proving Theorem 4.1 with these weaker conditions made the theorem stronger (accounting for a larger set of shapes). In particular, we can include convex shapes that adhere to the conditions provided in [3], and the theorem would still hold. We use this fact throughout, and after citing [3], this should be a triviality, but it is worth noting for completeness.*

5 Arguments with Colouring Numbers

The next two sections attempt to close our main goal, which is to show that we can find a set S of certain types of shapes in $\mathbb{R}^d$, not necessarily convex, which is the codomain of a function mapping vertices of an n vertex graph, and admits a c-thin intersection representation that can be separated by a set of size $O(n^{1-\frac{1}{2d+4}})$. To prove this, we require a bit of groundwork with colouring numbers.

5.1 Supporting Lemma

Lemma 5.1. *Let $A, B \subseteq \mathbb{R}^d$ be shapes such that B is centrally symmetric. Let*

$$\mathcal{B} = \bigcup_{B' = p_i + B} \{B' : B' \cap A \neq \phi\}$$

Where p_i is a translation vector taking B to B' for each B'. Then $\mathcal{B} \subseteq A + 2B$.

This lemma is accompanied by the picture in Figure 13. The picture depicts a centrally symmetric, non-convex shape B, and a non-convex shape A. The idea of the following proof is that since B is centrally symmetric, $-B = B$. Therefore, if we consider points in B', a translation of B such that $B' \cap A$ is nonempty, then the furthest away any point in such a B' can be is on the boundary depicted by a dashed line in the left side of Figure 13. Now, if we were to consider the union of all translations of B such that the translating vector is a point in A, then we would end up with the first dashed line we see on the right side of Figure 13. If we now draw more translates of B, centred on the aforementioned dashed line, one may notice that these translates intersect the shape A at the boundary, suggesting that the set $\mathcal{B}$ would be contained within the second larger boundary in the left side of Figure 13. This makes intuitive sense as we have essentially created a boundary large enough to fit a copy of B outside our shape A, but within the said boundary. Now let's consider a more rigorous proof.

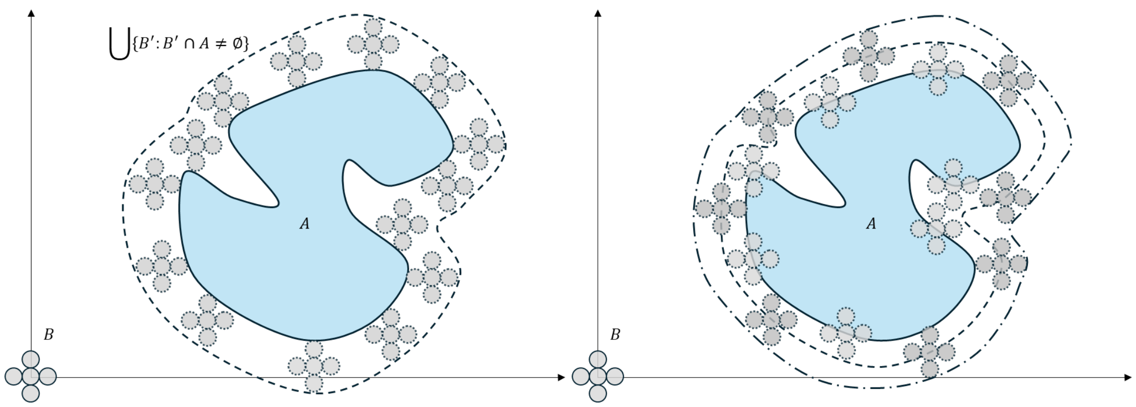

Figure 13: Picture Supporting the Proof of Lemma 5.1

Proof. Note that the statement in the lemma is equivalent to showing that for every $u \in \mathbb{R}^d$, if A and some translate of B, $u + B$ have non-empty intersection, then this same translate $u + B$ is a subset of $A + 2B$. In other words, it says that every element in the union is $A + 2B$, so the union is a subset of $A + 2B$. Therefore, u is a translation vector which takes at least one point in B to a point in A. Let $a \in A$ and $b \in B$ be such points. By construction, $u + \underline{b} = \underline{a}$ which implies that $u = \underline{a} - \underline{b}$ for any such a and b in the respective sets. This means that $u \in A + (-B)$ where $-B$ is the shape B reflected through the origin, since for any $b \in B$, $-b$ is just b reflected in the origin since B is centrally symmetric. Using a similar logic, since B is centrally symmetric, B reflected through the origin is in fact just B, that is that B is invariant under reflections through the origin as a result of it being centrally symmetric. So $B = -B$ and so $u \in A + B$. This means that $u + B \in A + 2B$ for all $u \in \mathbb{R}^d$ satisfying $(u + B) \cap A \neq \phi$. $\square$

This lemma's necessity and proof are inspired by [3] wherein the authors prove it specifically for convex shapes, and then use it for '*envelopes*' of convex sets which are parallelepipeds. These shapes are always convex and centrally symmetric. The same is true for this paper's notion of a tight box, which this paper applies Lemma 5.1 on. So strictly speaking, we do not need to generalise the above lemma for non-convexity, but the generalisation keeps in theme with the rest of the paper, and so it is included here.

5.2 Theorem Bounding The Generalised Strong Colouring Number With Respect to r and $\prec$

We can now state and prove one of the main arguments of [3] with respect to our carefully crafted non-convex sets, thanks to the previous chapter.

Theorem 5.1. *Let G be a graph on n labelled vertices, $\prec$ be a linear ordering of the numbers $\{1, 2, ..., n\}$, and thus an ordering of the vertices $v_1, v_2, ..., v_n$. Also let ϕ and ψ be functions assigning shapes in $\mathbb{R}^d$ to the vertices of G such that for all real numbers $c, s, k \geq 1$, and a natural number $d \in \mathbb{N}$, there exists a real $\delta \in \mathbb{R}$ such that:*

i For any vertex v in $V(G)$, $\psi(v)$ is a shape in our previously defined set of shapes, S, and $\phi(v) \subseteq \psi(v)$

ii ϕ is a c-thin function (recall that an intersection representation is c-thin if for all $x \in \mathbb{R}^d$, x belongs to at most c shapes that represent vertices of G).

iii For any $v, x \in V(G)$, if $v \prec x$, then $\psi(x) \leq_k \psi(v)$ and $\psi(x) \sqsubseteq_s \phi(v)$.

iv For any edge, $uv \in E(G)$, if $u \prec v$, then $\psi(v) \cap \phi(u)$ is non-empty

Then, $col_{\prec,r}(G) \leq \delta r^d$

Proof. Consider any $r \in \mathbb{N}, v \in V(G)$. We want to show that the size of the set $L_{G,\prec,r}$ is less that or equal to δr^d for any choice of r and v, as this would prove $\text{col}_{\prec,r}(G) = \max_{v \in V(G)} |L_{G,\prec,r}(v)| \leq \delta r^d$. Like in [3], we assume that $|L_{G,\prec,r}(v)| \geq 2$ since the case where $|L_{G,\prec,r}(v)| = 1$ is trivial; particularly, it would assert that there are no vertices less than or equal to a distance of r away that precede v according to $\prec$, since $v \in L_{G,\prec,r}(v)$, for a connected G.

We can start considering paths of length less than or equal to r from a chosen vertex v to an arbitrary vertex $x_m = u \prec v$ that respects the linear ordering $\prec$. Note that $u \neq v$ as a basic consequence of linear orderings. Let the path from v to u be denoted by $P_{vu} = vx_1x_2...x_{m-1}u$. Since it respects the ordering $\prec$, we can conclude that $v \prec x_1, ..., x_{m-1}$. Therefore, by *iii.* ($\psi(x_i) \leq_k \psi(v)$) and the fact that $\psi(v)$ is a shape in S, so has a tight box around it, $T(v)$, implies that $(\psi(x_i))' \subseteq k\psi(v) \subseteq kT(v)$, for all $i \in \{1, 2, ..., m\}$ and some translate of $\psi(x_i)$. Since our notation does not convey specific translations for each i, we will need to assign names to translations of the tight box around $\psi(v)$. Accordingly, let $T_0 = (T(v))'$, and $T_i = (kT(v))'$ such that $\psi(x_i) \subseteq (kT(v))$. Note that with this notation, $T(v)$ is centred at the origin, with T_i being some translation of the tight box around $\psi(x_i)$.

Now by *i.*, $\psi(v), \psi(x_i) \in S$ for all $i \in \{1, 2, ..., m-1\}$ and $\phi(v) \subseteq \psi(v)$ and $\phi(x_i) \subseteq \psi(x_i)$ for all $i \in \{1, 2, ..., m-1\}$. Crucially, if $\psi(x) \subseteq T_x$ for some vertex x and tight box T_x, then $\phi(x) \subseteq \psi(x) \subseteq T_x$. Now we can use *iv.* to prove that $T_i \cap T_{i-1}$ is a non-empty set. For the path $P_{vu} = v, x_1, ..., x_{m-1}, u$, any 2 vertices cannot have the same linear ordering by definition, so without loss of generality, assume that for $x_i x_{i-1} \in E(P_{vu})$, $x_{i-1} \prec x_i$. So by *iv.*, $\psi(x_i) \cap \phi(x_{i-1})$ is non-empty. $\psi(x_i) \subseteq T_i$ and $\phi(x_{i-1}) \subseteq \psi(x_{i-1}) \subseteq T_{i-1}$ as before, and so, $T_i \cap T_{i-1}$ is a non-empty set of all $i \in \{1, 2, ..., m-1\}$. For $i = m$, we know that that $x_m = u \prec v \prec x_{m-1}$, so by transitivity of linear orderings, $u \prec x_{m-1}$. So $\psi(x_{m-1}) \cap \phi(u)$ is a non-empty set contained in $T_{m-1} \cap \phi(u)$, which is consequently also non-empty. Consider any point in $T_{m-1} \cap \phi(u)$, and call it y_u

Since $u \prec v$, by *iii.*, we have that $\psi(v) \sqsubseteq_s \phi(u)$. which by definition states that for every point in $\phi(u)$, there exists a translation of $\psi(v)$, call it $(\psi(v)')$, which contains that point, and $\text{vol}(\phi(u) \cap \psi(v)') \geq \frac{1}{s}\text{vol}(\psi(v)') = \frac{1}{s}\text{vol}\psi(v)$. We will want to use these properties with y_u; note that since $y_u \in T_{m-1} \cap \phi(u) \subseteq \phi(u)$, $y_u \in \phi(u)$, and so we can find some translation of $\psi(v)$ which contains y_u. We will borrow notation from [3] which defines this translation of $\psi(v)$ (one which contains y_u) as B_u, to define $B_u^\star = B_u \cap \phi(u)$. We can clearly see that $B_u^\star$ must be contained in the union of all translations of $\psi(v)$ that have non-empty intersections with $\phi(u)$ $\left(B_u^\star \subseteq \bigcup_{y_z \in \phi(u)} \{B_z : (B_z \cap \phi(u)) \neq \emptyset\}\right)$.

Now we note that all the tight boxes we talk about are centrally symmetric, and so we can apply Lemma 5.1 whenever there is a non-empty intersection between the sets A and T_i. We will divide the work up into $r-1$ steps, considering the $r-1$ edges (and non-empty intersections) that we know are in the path. The first step is to consider $\phi(u) \cap T_{m-1}$, which we know has non-empty intersection as a result of the fact that $\psi(x_{m-1}) \subseteq T_{m-1}$ and $\psi(x_{m-1}) \cap \phi(u) \neq \emptyset$. Let $\psi(x_{m-1}) = A_0$. We want to consider the union of any translation of T_{m-1} that may have a non-empty intersection with $\psi(x_{m-1}) = A_0$, and so we consider $A_1 = A_0 + 2T_{m-1}$ which must contain this union of translations by Lemma 5.1. The next step is to consider $A_1 \cap T_{m-2}$, which we know also has non-empty intersection since we know that $\phi(x_{m-1}) \subseteq \psi(x_{m-1}) \subseteq T_{m-1} \subseteq A_1$ Once again we want to consider the union of any translation of T_{m-2} that has non-empty intersection with A_1, and so we apply Lemma 5.1 to get $A_2 = A_0 + 2T_{m-1} + 2T_{m-2}$. By repeatedly applying lemma 5.1 a maximum of $r-1$, acknowledging the fact that $T_i = (kT(v))'$ gives the following:

$$B_u^\star \subseteq \bigcup_{y_z \in \phi(u)} \{B_z : (B_z \cap \phi(u)) \neq \emptyset\} \subseteq (3 + 2(r-1)k)T(v) + p(v) = Y_v$$

for the translation vector $p(v)$ taking the origin to the centre of $\psi(v)$. Simply, $B_u^\star$ is contained within the boundary of the shape Y_v, which is centred at v, by simply accounting for every translation of every intersecting tight box, which represents an edge in the graph.

After noting that our choice of $B_u^\star$ forces it to be contained in $\phi(u)$ by construction, let $X_v =$

$\bigcup_{u \in L_{G,\prec,r}(v) \setminus \{v\}} B_u^\star$. Using the definition of $\sqsubseteq_s$, *ii.* and relations from *iii.*, we get:

$$\text{vol}(X_v) \geq \frac{1}{c} \sum_{u \in L_{G,\prec,r}(v) \setminus \{v\}} \text{vol}(B_u^\star) \geq \frac{|L_{G,\prec,r}(v)| - 1}{sc} \text{vol}(\psi(v)) \geq \frac{|L_{G,\prec,r}(v)|}{2sc} \text{vol}(\psi(v))$$

Also, X_v is a union of $B_u^\star$'s for all values of u which necessarily have non-empty intersection with $\phi(u)$, and so it is contained within Y_v. Therefore, with the above and by applying Lemma 4.2, we get the following:

$$\text{vol}(X_v) \leq \text{vol}(Y_v) \leq (3 + 2(r-1)k)^d \text{vol}(T(v)) \leq \left(\frac{3}{r} + 2k\right)^d r^d (pd)^d \text{vol}(\psi(v))$$

So, putting the above two inequalities together and noting that $r \geq 1$, we get:

$$\frac{|L_{G,\prec,r}(v)|}{2sc} \text{vol}(\psi(v)) \leq \text{vol}(X_v) \leq (3 + 2k)^d r^d p^d d^d \text{vol}(\psi(v))$$

Which implies:

$$|L_{G,\prec,r}(v)| \leq 2sc(3 + 2k)^d r^d p^d d^d$$

So setting $\delta = 2sc(3 + 2k)^d p^d d^d$ completes the proof. □

The article [3] claims that for convex shapes $\delta = 2sc(2k+1)^d d^d$ instead of $\delta = 2sc(2k+3)^d d^d$, which would follow from the working it provides. Because I am not confident in how [3] reached this better bound, I do not claim that the result in Theorem 5.1 is optimal, just that there is a value of δ for which the theorem holds, which for our purposes is more than enough.

We want to use Theorem 5.1 with a theorem relating the colouring number and the existence of a separator with sublinear size. This is out of the scope of this essay, but can be seen as a result of Lemma 2 in [2] and Observation 10 [5], as cited in [3]. Such a theorem is stated below:

Theorem 5.2. *For all real numbers $a, p > 0$, there exists a real number γ such that if a graph G satisfies $col_r(G) \leq ar^p$ for every natural number r, then G has a (balanced) separator of size at most $\gamma n^{1 - \frac{1}{2p+4}}$*

6 Putting it all Together

The main theorem we are trying to prove is a modification of Theorem 2 in [3], accounting for the possibility of certain non-convex shapes in the set S, as defined earlier. This modified theorem is stated below.

Theorem 6.1. *For all real numbers $c, s \geq 1$, and natural numbers d, there exists a real β such that every n-vertex graph admitting a $(c, \sqsubseteq_s, S)$-tame representation in $\mathbb{R}^d$ has a (balanced) separator of size at most $\beta n^{1 - \frac{1}{2d+4}}$*

The proof for this theorem follows almost trivially from Theorem 5.2 and the theorem stated below, which describes the relationship between graphs that admit $(c, \sqsubseteq_s, S)$-tame representations and the strong coloring number of such graphs with respect to a linear ordering $\prec$.

Theorem 6.2. *For all real numbers $c, s \geq 1$ and natural numbers d, let G be a graph that admits a $(c, \sqsubseteq_s, S)$-tame representation, φ, in $\mathbb{R}^d$. Also, let $\prec$ be a linear ordering which orders vertices, v, in $V(G)$ in decreasing order according to vol$(\varphi(v))$. Then col$_{\prec,r}(G) \leq \delta r^d$ for all natural numbers, r.*

And so after proving Theorem 6.2, we would have achieved the goal of this essay: to prove that certain non-convex intersection representations have balanced separators. All that remains now is to prove the existence of, and find the relationships between, $\leq_{k,s}$ comparability and $\sqsubseteq_s$ comparability, which will allow us to prove Theorem 6.2, closing all our goals.

6.1 Relationships Between $\leq$, $\sqsubseteq$ and $\prec_{\text{vol}}$

We start with a lemma relating $\leq_{k_1,k_2}$ and $\sqsubseteq_s$.

Lemma 6.1. *Let B_1 and B_2 be shapes in $\mathbb{R}^d$, and let k, s be real numbers such that $k_1, k_2, s \geq 1$. if $B_1 \leq_{k_1,k_2} B_2$, then $B_1 \sqsubseteq_s B_2$ for $s = (max(k_1, k_2))^d$*

Proof. We assume that $B_1 \leq_{k_1,k_2} B_2$ and use the definition to find the value of s for which $B_1 \sqsubseteq_s B_2$. For any point x in B_2, we want to consider a translation of B_1 which contains $\frac{1}{s}x$. We will assume that the point x is the origin without loss of generality, since we want to make a homothetic argument. Accordingly, we can see that $x \in k_1 B_2$ and so, by definition of $\leq_{k_1,k_2}$, $x \in k_2 B_1'$ and the translation of B_1, B_1' is contained within the set $k_1 B_2 \cap k_2 B_1'$. In actuality, $x \in B_1'$ as the origin is the centre of our homothetic scaling, which will be needed later. This containment implies that $\text{vol}(B_1) \leq \text{vol}(k_1 B_2 \cap k_2 B_1')$.

Let $s' = \max(k_1, k_2)$. Now we have that $\frac{k_2}{s'} B_1' \subseteq B_1'$ and $\frac{k_1}{s'} B_2 \subseteq B_2$, which implies that $\text{vol}\left(\frac{k_2}{s'} B_1'\right) \leq \text{vol}(B_1')$ and $\text{vol}\left(\frac{k_1}{s'} B_2\right) \leq \text{vol}(B_2)$, implying that $\text{vol}\left(\frac{k_1}{s'} B_2 \cap \frac{k_2}{s'} B_1'\right) \leq \text{vol}(B_2 \cap B_1')$. By noting that the contraction of a shape in d dimensions by a factor of x leads to a reduction in volume of that shape by a factor of x^d, we can contract $\left(\frac{k_1}{s'} B_2 \cap \frac{k_2}{s'} B_1'\right)$ by a factor of s' to conclude that $\text{vol}\left(\frac{k_1}{s'} B_2 \cap \frac{k_2}{s'} B_1'\right) = \frac{1}{(s')^d} \text{vol}\left(k_1 B_2 \cap k_2 B_1'\right)$. Therefore:

$$\text{vol}(B_1' \cap B_2) \geq \text{vol}\left(\frac{k_1}{s'} B_2 \cap \frac{k_2}{s'} B_1'\right) = \frac{1}{(s')^d} \text{vol}\left(k_1 B_2 \cap k_2 B_1'\right) \geq \frac{1}{(s')^d} \text{vol}(B_2 \cap B_1')$$

So by setting $s = (s')^d$ and noting from earlier that the origin is in our translated B_1, we can conclude that $B_1 \sqsubseteq_s B_2$. $\square$

As is the case in [3], we will need to consider $\gamma_d > 0$ from some positive integer d, which we will define as the maximum $(d-1)$-dimensional volume of an intersection of a hyperplane with a unit

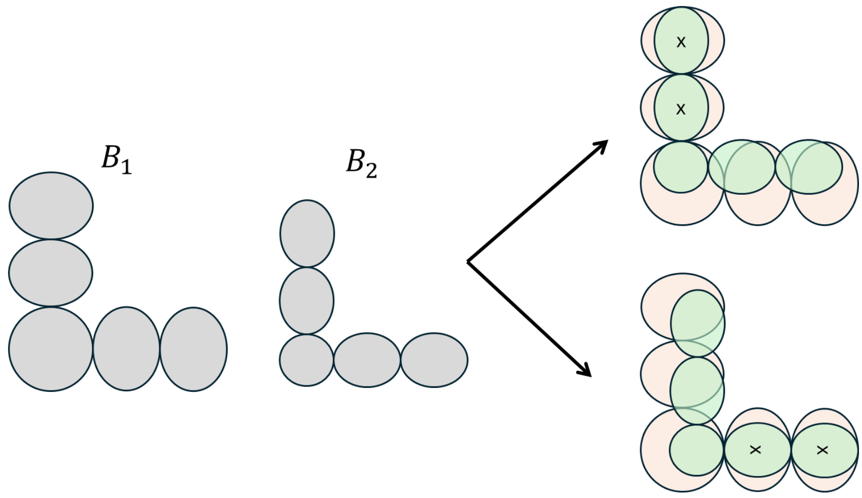

Figure 14: Examples of B_1 and B_2 in $B_1' \cap B_2$

cube in $\mathbb{R}^d$. Note that this definition is a direct quote from [3]. We will use it to show some kind of equivalence between $\sqsubseteq_s$, $\leq_{k,k}$ and the volumes of shapes.

Lemma 6.2. *Let B_1 and B_2 be shapes in S, and let $s \geq 1$ be a real number. Let $k = sp^{d+1}d^{d+\frac{3}{2}}\gamma_d$. If $B_1 \sqsubseteq_s B_2$, then $B_1 \leq_{k,k} B_2$*

At the introduction of this lemma, we encounter the first limitation of this approach. The definitions require us to consider shapes $B_1' \cap B_2$ for a specifically translated B_1, the issue being that in the general case, for a given s and k positive real numbers greater than 1, it is not necessary that we find a translation of B_1 such that the shape $B_1' \cap B_2$ is in the set S, has generalisable height, or even is connected for that matter. [3] is able to use such a lemma as a result of the fact that the intersection of 2 convex shapes is a convex shape.

There are many ways to overcome this limitation. For this essay we won't consider perhaps intuitive cases like fixing b and varying a in our construction of the disk packings, placing a bound over p so that the shapes don't get arbitrarily large. These could be effective if we set up the paper differently, but since all our theorems require a constant p value, we can't employ this method. However, I have still detailed it so that I can refer to it in later sections:

Assume we construct S' by fixing b and varying a, and placing bound on p. Now we can see that for any chosen point in B_2, if $B_1 \sqsubseteq_s B_2$, then this point is in a translation of B_1. The volume of $B_1 \sqsubseteq_s B_2$ being greater than $\frac{1}{s}$ times the volume of B_1 is another restriction on how big B_1 can get, effectively bounding a again. This concept is pictured in Figure 14. Note that in this example, we will always have $B_1' \cap B_2$ has height a_{B_2}, which is what is most appealing about this approach

For our purposes, we need to consider a method to maintain a constant p value between all our shapes in S' and solve the aforementioned issue of translation. This can be done by noticing the symmetry in our shapes. We will have to choose s such that the volume condition is satisfied in the definition of $\sqsubseteq_s$. This means that the amount by which shapes in S' can be scaled in relation

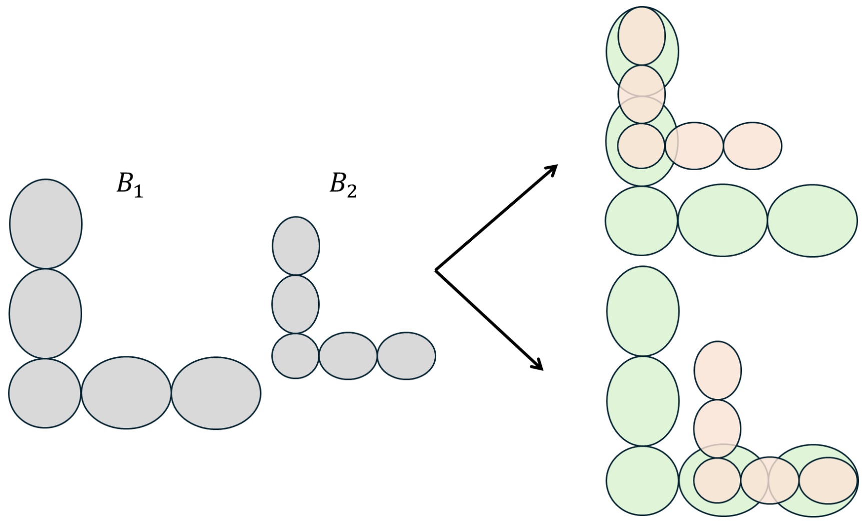

Figure 15: Examples of B_1 and B_2 in $B_1' \cap B_2$

to each other will be limited because of how inefficiently we let B_1 and B_2 intersect each other. However, since in this paper we are looking for a sufficient condition to construct graphs with sublinear separators, we can simply choose our s to be large enough to satisfy the shapes we want to be $\sqsubseteq_s$ comparable. This is not as elegant a solution as the previous paragraph, but it fixes the issue of variable ratios between the length and height of a shape.

Proof of Lemma 6.2. By applying a bijective affine transformation, we can assume that the unit cube, T_1, is a tight box around B_1, as we have been doing in previous proofs. This implies that $\text{vol}(B_1) \leq 1 = \text{vol}(T_1)$. By Lemma 4.2 we know that $\frac{1}{pd}T_1 \subseteq B_1$ and so, $\text{vol}(B_1) \geq \left(\frac{1}{pd}\right)^d$. Consider any point in kB_2, and without loss of generality, translate that point to the origin. Since $B_1 \sqsubseteq_s B_2$, we know that the origin is in sB_1', which implies that the origin is also in B_1', and we know that $\text{vol}(B_1' \cap B_2) \geq \frac{1}{s}\text{vol}(B_1) \geq \frac{1}{s}(pd)^d$. Now consider the tight box around $B_1' \cap B_2$, and say it has height h_T'. The $B_1' \cap B_2$ has volume at most $\gamma_d h_T'$. Therefore $B_1' \cap B_2$ has height at least $\frac{1}{sp^d d^d \gamma_d}$ and so by Lemma 4.6, $B_1' \cap B_2$ contains a ball of diameter $\frac{1}{sp^{d+1} d^{d+1} \gamma_d}$. Since B_1 is contained within a ball of diameter $\sqrt{d}$, we can see that $sp^{d+1} d^{d+\frac{3}{2}} \gamma_d (B_1' \cap B_2)$ contains B_1', and so $k(B_1' \cap B_2) = (kB_1)' \cap kB_2$ contains B_1'. We also know from the first step. that the origin is in $(kB_1)'$. This implies that $B_1 \leq_{k,k} B_2$, which is our goal □

Remark 6.3. *In the proof of the above lemma, we used disk packings of balls as our shapes in S. The proof can very easily be adapted for 2-thin neighbourhood sets constructed with ball-like shapes, by adjusting the formula for p accordingly. This step is fairly simple yet inconvenient to write out, especially when all we need to take away from it is that this height is indeed bounded. Thus, for brevity, I have omitted this part of the proof. However, if one were to complete it, it would work the exact same as the one above, just with a different set S' of neighbourhood sets with fixed b, varying a, and fixed intersection length.*

From the above remark, it is clear that one has a variety of choices when it comes to picking a set S to work in. For completeness, we will be working with the notions we have formally proven, however, the remark lays out how one would prove it, should they want to consider a non-zero intersection length.

The final thing we need to show is a relationship between comparability and the volume of our shapes.

Lemma 6.4. *For every real number $s \geq 1$ and natural number d, there exists a real number $s' \geq s$ such that if B_1 and B_2 are $\sqsubseteq_s$-comparable shapes in S, and vol$(B_1) \leq$ vol(B_2), then $B_1 \sqsubseteq_{s'} B_2$*

Proof. Let k be as it was in Lemma 6.2. Since B_1 and B_2 are $\sqsubseteq_s$-comparable, we consider both cases. If $B_1 \sqsubseteq_s B_2$, then we can simply set $s' = s$ and conclude since $B_1 \sqsubseteq_s B_2$ implies $B_1 \sqsubseteq_{s'} B_2$. Now consider the other case, that $B_2 \sqsubseteq_s B_1$. By Lemma 6.2, we know that this implies $B_2 \leq_{k,k} B_1$. By choosing the origin again, without loss of generality, this means that $B_2' \subseteq kB_2' \cap kB_1$ which implies that $\frac{1}{k}B_2' \subseteq B_2' \cap B_1 \subseteq B_1$ which implies that $B_2' \subseteq kB_1$, or that $B_2 \leq_k B_1$. This means that if $k \leq 1$, we have that $B_1 \sqsubseteq_s B_2$. So now assume that $k > 1$. Again $B_2' \subseteq kB_2'$ since the origin is in B_2'. Therefore, we can conclude that $\text{vol}(B_2' \cap B_1) = \frac{1}{k^d}\text{vol}(kB_2' \cap kB_1) \geq \frac{1}{k^d}\text{vol}(B_2' \cap kB_1) = \frac{1}{k^d}\text{vol}(B_2') = \frac{1}{k^d}\text{vol}(B_2) \geq \frac{1}{k^d}\text{vol}(B_1)$. This implies that $B_1 \sqsubseteq_{k^d} B_2$. By choosing $s' = \max(s, k^d)$, we can ensure that there exists an $s' \geq s$ such that $B_1 \sqsubseteq_{s'} B_2$ □

6.2 Proof of the Main Theorem

Now we have all the tools to be able to prove Theorem 6.2.

Proof of Theorem 6.2. It suffices to show that the conditions of Theorem 5.1 are met, as that would imply that $\text{col}_{\prec,r}(G) \leq \delta r^d$ for $\delta = 2sc(3 + 2k)^d p^d d^d$. let $\phi = \psi = \varphi$, which is a c-thin intersection representation taking vertices in $V(G)$ to shapes in S' where S' is the set of shapes with all the restrictions we have put on it so far in the paper. Clearly $S' \subseteq S$ since we have taken out a lot of shapes in S to make S', and so condition *(i)* is satisfied. Conditions *(ii)* and *(iv)* are also satisfied by construction. So we are left to prove condition *(iii)*. Let s' and k' be real numbers greater than 1. Note that if $u \prec v$, then $\text{vol}(\psi(v)) \leq \text{vol}(\psi(u))$. Therefore by Lemma 6.4, $\psi(v) = \varphi(v) \sqsubseteq_{s'} \varphi(u) = \phi(u)$. Also by Lemma 6.2, we have that $\psi(v) = \varphi(v) \leq_{k'} \varphi(u) = \psi(u)$. This holds for our choices of k' and s', which were $sp^{d+1}d^{d+\frac{3}{2}}\gamma_d$ and $\max(s, k'^d)$ respectively. Note that the variables addressed in all conditions only depend on c, s and k, while everything else, like p and d, are fixed values we determine before we construct a graph. This construction means *(iii)* also holds, and so Theorem 6.2 holds. □

As stated earlier, Theorem 6.1, which is the main goal of this paper, follows trivially from previous results.

7 Conclusion

In this paper, we were able to generalise many results pertaining to convex shapes, to a union of convex shapes that together are not necessarily convex. We showed that if we can create an intersection representation of a graph with shapes from S', that is $(c, \sqsubseteq_s, S')$-tame in $\mathbb{R}^d$, then we can find a balanced separator of order $O(n^{1-\frac{1}{2d+4}})$. We came to this conclusion through 3 main steps.

As laid out in the introduction, there were 2 key problems with intersection representations. If we consider non-convex shapes, then we can make arbitrarily large cliques. Similarly, if we allow the use of sufficiently non-similar shapes, then we allow the creation of arbitrarily large complete bipartite graphs.

We solved these issues by considering specific types of non-convex shapes. The first issue was described and solved by an example. Here I would like to describe another way of viewing the problem. Since the shapes we consider are non-convex, we know that there are 2 points, x and y, in the shape such that a straight line from one to the other, L_{xy}, crosses the boundary of the shape. Because we had no restrictions (yet) on the thickness of the shape, we were allowed to make such non-convex shapes arbitrarily thin. This means that there is an arbitrarily large number of sets that can fit in the gap where the L_{xy} is not in the shape. So, for an arbitrarily small thickness and an arbitrarily small translation, we can fit an arbitrarily large number of shapes that all intersect each other. To overcome this, we simply need to disallow the arbitrarily thin shapes, which was done by imposing a bound on the ratio between the diameter of the smallest ball containing the shape, and the largest ball contained within the shape.

The second issue was not an issue of non-convexity but more an issue of shapes that lack symmetry. The idea was that if we take a long rectangle and rotate it $45°$, then the shapes are dissimilar enough such that duplicates can create complete bipartite graphs. The solution to this issue was to refine the choices of shapes we considered. Particularly, we consider a bounded non-convex set constructed by a union of 5 ball-like shapes, arranged in quite a specific way, with the key being the pivot ball: a ball to which we appended 2 pairs of ball-like shapes. Having this feature allowed us to scale a tight box around the whole shape, to an envelope around the pivot ball, which in turn allowed us to use convexity to prove various lemmas, leading up to the main theorem of Section 4, Theorem 4.1. The theorem itself uses the lemmas to create 2 sets of m shapes, $L_1, ..., L_m, R_1, ..., R_m$ wherein all L and R shapes are disjoint from each other but for any i and j in $1, ..., m$, $L_i \cap R_j$ is a non-empty set. This theorem is how we can rigorously prove that in a set of shapes, S, if we hope to make an intersection representation of a graph that admits balanced separators, we certainly can't have two shapes that are not $\leq_k$-comparable, for $k = 2p^3 d^{\frac{5}{2}m}$. Or in other words, all shapes in S must be pairwise $\leq_k$-comparable, otherwise we can make the complete bipartite graph $K_{m,m}$ with duplicates of the 2 $\leq_k$-non-comparable shapes as shown in the theorem.

We also stated and proved some corollaries which can extend the theorem stating the necessity of $\leq_k$-comparability to shapes made solely of five balls, or an odd number of balls.

We then opened the discussion on colouring numbers in hopes of proving the conditions that need to be satisfied by an intersection representation if we want to produce a graph which has a generalised strong colouring number less than δr^d for some real δ. We are interested in this because Theorem 5.2 (which states that a graph with generalised strong colouring number at most ar^p, must have a balanced separator) can be seen as a direct result of sources [2] and [5]. Under the 4 stated conditions, we can find a path of length $m \leq r$, from v to u, that respects some linear ordering $\prec$ and can be represented by the intersection representation $\psi(P_{xy})$. The last edge in such a path is from a vertex that precedes v, to one that succeeds it. This means that the tight box around $\psi(v)$ and the shape $\phi(u)$ have a non-empty intersection. We use this fact to find a translation of $\psi(v)$ which contains a point in this intersection. By noting that there are $r-1$ edges between v and u, and the fourth condition in the statement of the proof, we can see that we can repeatedly apply Lemma 5.1 a maximum of $r-1$ times to a tight box around $\psi(v)$ to construct Y_v. We also construct X_v as a union of non-empty intersections of B_{x_i} with $\phi(x_i)$, for $i \neq 0$. Thus we get an upper and lower bound for X_v which both depend on constants, $|L_{G,\prec,r}(v)|$ and $\text{vol}(\psi(v))$, the latter of which gets cancelled out in the balancing. Thus, we get an upper bound for $|L_{G,\prec,r}(v)|$ as a single-term function of r^d with its coefficient dependent on constants.

Following results from the sources, in tandem with this result, show that it is sufficient for an intersection representation to satisfy these conditions in order to represent a graph that has a balanced separator, which is the overall goal of the essay. The reason we weren't done yet was because we didn't have a way of relating volume of shapes, which is the linear order we would like to impose on the vertices of G, to $\sqsubseteq_s$ comparability. Section 6 resolves this but needs to make quite a large concession in order to do so. Nevertheless, as a result of the work in this paper, we can construct intersection representations with certain non-convex shapes to find graphs that admit balanced separators.

8 Open Questions and Extensions

8.1 Extensions

The most obvious extension to the work done in this paper would be to generalise the non-convex shapes we consider further. It may be possible to consider arbitrarily dense packings of ball-like shapes with the necessary bounds on things like aspect ratio implemented, so that each shape being considered can look like an L-shape. Such a case is depicted in Figure 16. Note that repeatedly doing this for shapes with larger values of a and smaller values of b can give shapes that look more like the L-shapes we talked about in the introduction. Alternatively, we can try to consider appending rectangles at the ends of the shapes.

We should also consider a fairly simple extension of this paper, which is that some unions of balls in different orientations would follow almost identically to what we have proven. While researching for and writing this paper, I frequently thought about such shapes but refrained from talking about them in the main body of the essay so as to keep the focus of the discussion theory-based, while

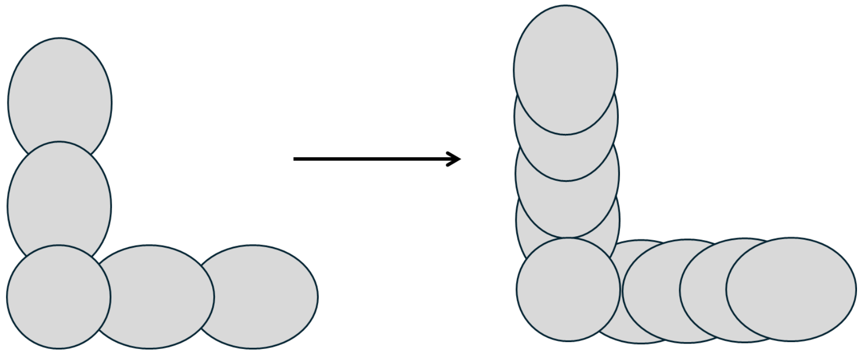

Figure 16: Example of a Transformation of Shapes in S'

not having to repeat arguments that were made with slight adjustments. Examples of such shapes constructed using balls are depicted in Figure 17, and their counterparts constructed with ball-like shapes are depicted in Figure 18. The important thing to note in these examples is that all of the constructed shapes have a pivot ball of size a, and so the notion of and the theorems pertaining to tight boxes still hold relevance over these shapes.

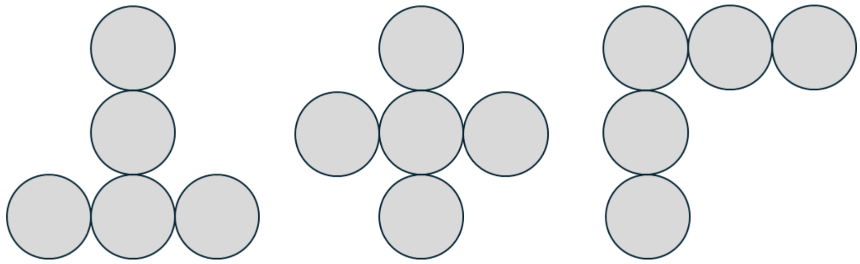

Figure 17: Other Shapes that could be considered for S'

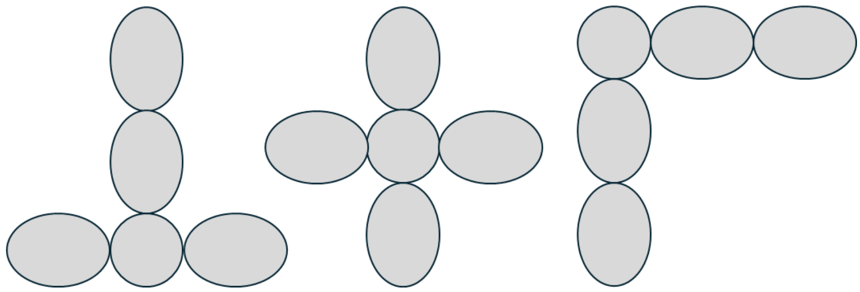

Figure 18: Other Shapes that could be considered for S'

In fact, we can now state the following conjecture.

Conjecture 8.1. *Any* $90°$ *rotation of the 3 shapes seen in Figure 17 or Figure 18 can be used as shapes in* S' *in Theorem 6.1 (The main theorem of this paper)*

Notice that our value for p will remain fixed even if we consider all 3 of these classes of shapes. This means that the classes all adhere to the same values of p, k, s, etc., defined in the various proofs throughout this paper, suggesting that the only thing missing for the conjecture to be true is some kind of translation which ensures that pivot circles are contained within each other, perhaps by a bijective affine transformation of tight boxes of one of the classes of shapes.

The last extension I think is worth mentioning is the concept of tight rectangular boxes. Throughout this paper, we consider balls appended to 2 axes, d_1 and d_2, and so the tight box in that plane is a square since the lengths of the largest line segments parallel to both those dimensions are equal. Still, we can expect the majority of the theorems to still hold true for such a shape, B, by taking $p = \max(p_1, p_2)$ for p_1 and p_2 the largest line segment contained within the shape parallel to the axes d_1 and d_2 respectively, to ensure that a tight box T satisfies $T \leq_{pd} B$. Examples of such shapes are depicted in Figure 19. However, we may run into issues again in Lemma 6.2 when considering the heights of intersections of shapes. Nevertheless, with the context of this research project, this does not seem like an unattainable inference, up to some kind of restriction of the set S that we consider.

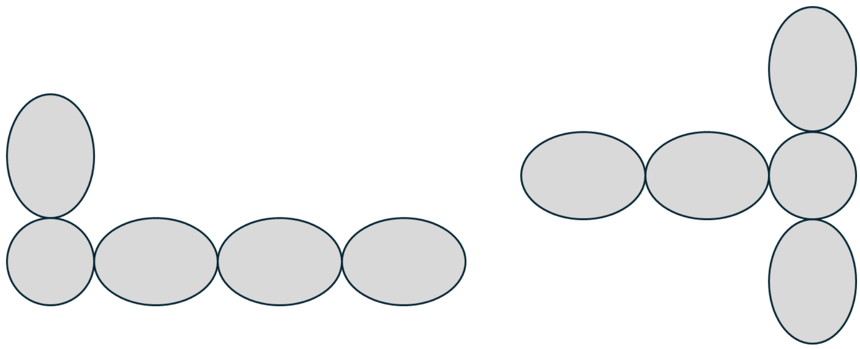

Figure 19: Examples of Shapes that have Rectangular Tight Boxes in the d_1, d_2-plane

8.2 Open Questions

There are a series of open questions that I found were out of the scope of this essay, but could be relevant to the subjects being talked about. These are listed below.

Most relevantly, $\sqsubseteq_s$ is not a necessary condition for a c-thin neighbourhood system to represent a graph which has a balanced separator. The authors of [3] concede this for convex shapes, giving an example in their **Concluding Remarks**. It remains an open question whether such a necessary condition exists for both convex and non-convex shapes.

Something I would spend longer on, given the time, is the concessions that were made in the proof of Lemma 6.2. We heavily restrict the kinds of shapes we can consider at once, giving ourselves only 1 degree of freedom to create new shapes, with the imposition of a fixed value for p. We already talked about how fixing b seemed somewhat cleaner, but ended up disproving large chunks of previous lemmas which makes the paper obsolete. However, I was optimistic that there was some way to characterise the diameter of the smallest ball contained in $B_1' \cap B_2$, albeit in a messy way, to generalise the lemma. Since the goal of this essay is simply a preliminary probing of non-convex sets in intersection representations, I thought it best not to diverge from the goal too much, given the scope of this essay within the time frame.

Notably, our proofs relied on the fact that our shapes consisted of pivot balls. As a result, we knew that by construction, the largest diameter of a ball contained within our shape was of size a, which is a fraction of the largest line segment contained within the shape. And so, an open question posed by the paper is how one can generalise the theorems proven in the paper using shapes that don't have this nice relationship between the largest circle contained within the shape and the largest line segment contained within the shape

Lastly, the question about non-convex shapes in general can be asked. If we make concessions like bounding the aspect ratio, fixing a circle of size at least l within the shape, fixing a ratio of the largest line segment against l like we did for p throughout this paper, and any other conditions we might need, is it possible to extend the theories presented in this paper to arbitrarily chosen convex shapes that contain such a circle? Such an extension seems somewhat natural, and if generalisable, means we can use a wide variety of shapes like those depicted in Figure 20 and more, to represent vertices in intersection representations.

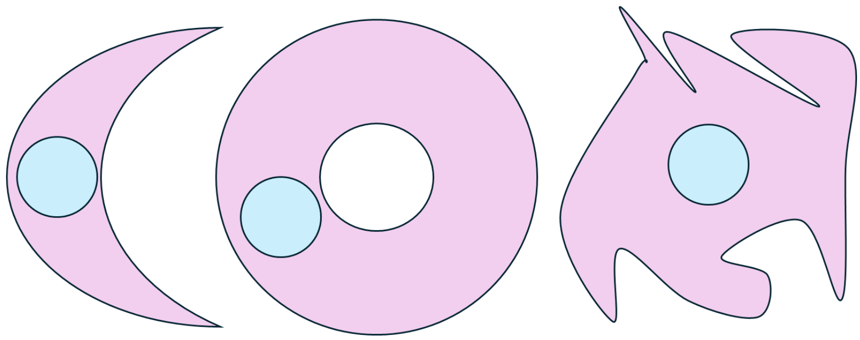

Figure 20: Non-Convex Shapes that contain a circle of diameter l

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