Group Walk Random Graphs and Benjamini-Schramm convergence

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Abstract

This paper is divided in two mostly independent parts that both explore a different aspect of random walk and random graphs. We first study the Benjamini-Schramm convergence and we prove the recurrence of certain graphs that are Benjamini-Schramm limits. The second part is the study of a specific random graph. Most of the results showed in this part are new and due to myself.

Contents

Introduction

A graph G is defined by a set of vertices V and a set of edges $E \subseteq V \times V$. We say that two vertices u, v are neighbours iff $(u, v) \in E$. The degree of a vertex is the number of neighbours it has. The graph we consider are of finite degree. We work on non oriented simple graphs, meaning that no vertex is linked to itself and that E is symmetric. We sometimes confuse G with its own set of vertices. In the first section, we also assume that graphs are simply connected, meaning removing one edge will not make it disconnected.

One of the main topics in this paper are simple random walks on graphs. A simple random walk on a graph is a Markov process $(X_n)_{n\in\mathbb{N}}$ whose states are the vertices of the graph and whose probability of transition is defined as follows: for all u, v in V, $\mathbb{P}(X_n = u|X_{n-1} = v) = \frac{1}{\deg(v)}$ if there is an edge between u and v, 0 otherwise.

In the first section of this work, we prove a recurrence result on certain random graphs. The proof is taken from [BS01]. The second section is mostly the extent of the works of Agelos Georgakopoulos and John Haslegrave in [GH16].

1 Benjamini-Schramm convergence

1.1 Definitions and examples

This approch of the Benjamini-Schramm convergence is largely inspred by [ATV].

We work in the space $\mathcal{RG}_D$ of the rooted connected simple graphs with degree bounded by $D \in \mathbb{N}$. A root is simply a distinguished vertex. We can put a distance on that space: let (G_1, o_1) and (G_2, o_2) in $\mathcal{RG}_D$, and d the largest integer such that $B_{G_1}(o_1, d)$ is isomorphic to $B_{G_2}(o_2, d)$, then $d((G_1, o_1), (G_2, o_2)) = \frac{1}{d}$. By $B_G(o, d)$ we mean the subgraph of G whose vertices are vertices of G at a combinatoric distance to o smaller than d.

A random rooted graph is therefore a Borel probability distribution on $\mathcal{RG}_D$. A sequence of random graphs converges iif the sequence of distributions converges weakly.

Definition. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability distributions converges to μ iif for all continuous function $f : \mathcal{RG}_D \to \mathbb{R}$, $\int_{\mathcal{RG}_D} f(x)d\mu_n(x)$ tends to $\int_{\mathcal{RG}_D} f(x)d\mu(x)$*

Since the neighbourhoods of the root constitute a clopen base of $\mathcal{RG}_D$, we can say that the weak convergence is equivalent to the convergence of the measures of the base sets. By neighbourhoods of the root we mean the sets $\{G \in \mathcal{RG}_D | B_G(o, d) = H\}$ for any $d \in \mathbb{N}$ and $H \in \mathcal{RG}_D$. The precedent definition is therefore equivalent to :

Definition. *A sequence $((G_n, o_n))_{n\in\mathbb{N}}$ of random rooted graph converges in the sense of **Benjamini-Schramm** to (G, o) iif for all $k \in \mathbb{N}^*$ and $\alpha \in \mathcal{G}_D$ finite, $\mathbb{P}(G_n \cap B(o_n, k) \cong \alpha)$ converges to $\mathbb{P}(G \cap B(o, k) \cong \alpha)$.*

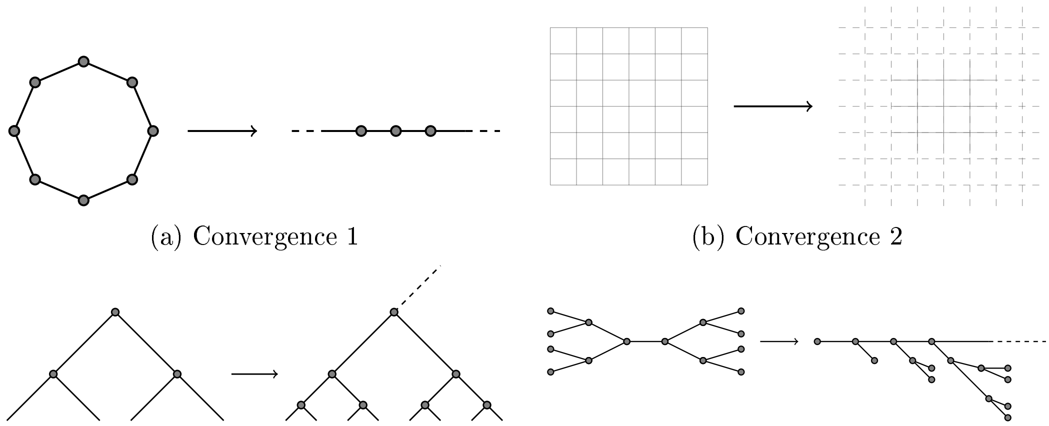
Example. *The n-cycle, where the root is chosen uniformly, converges to a graph a.s. equal to $(\mathbb{Z}, 0)$.*

The $n \times n$ grid, where the root is chosen uniformly, converges to a graph a.s. equal to $(\mathbb{Z}^2, (0, 0))$.

The binary tree of height n where the root is chosen uniformly among the leaves converges to a graph a.s. equal to the canopy tree (the root being any of the leaves).

The binary tree of height n where the root is chosen uniformly among the vertices converges to a graph a.s. equal to the canopy tree (the root being of height h with probability 2^{-h} (see section 2 for more details)).

The 3*-regular graph of size* n *where the root is chosen uniformly among the vertices, does not converge to the* 3*-regular graph but to the canopy tree.*



(a) Convergence 1

(b) Convergence 2

(c) Convergence 3

(d) Convergence 4

Figure 1: Examples

1.2 Recurrence

In this section, we present the proof by Benjamini and Schramm that certain limits of graphs are recurrent.

We say that a graph is recurrent iff any simple random walk a.s. returns to its point of origin an infinite number of times.

In order to present the theorem, we need to define the notion of unbiased random graphs. If H is a finite graph with degree bounded by $D \in \mathbb{N}$, we take μ_H as the probability measure such that for all A in Borel subset of $\mathcal{RG}_D$, $\mu_H(A)$ is the probability that $(H, o) \in A$ when o is chosen uniformly among the vertices of H.

Definition. *A finite random rooted graph is said* ***unbiased*** *if its distribution is in the closed convex hull of the measures* μ_H*.*

To put it in simply, a random rooted graph (G, o) is unbiased if for all H finite, conditioned to the event $\{G = H\}$, o is chosen uniformly among the vertices of H.

We can now express the main result of this section:

Theorem 1. *[BS01] Let* (G, o) *be the Benjamini-Schramm limit of a sequence* $(G_n)_{n\in\mathbb{N}}$ *of unbiased, finite, planar, simply connected graphs in* $\mathcal{RG}_D$ *for some* $D \in \mathbb{N}$*, then a.s.* G *is recurrent.*

In order to prove the theorem, we will prove:

Theorem 2. *[BS01] Let* (G, o) *be the Benjamini-Schramm limit of a sequence* $(G_n)_{n\in\mathbb{N}}$ *of unbiased, finite triangulations of the sphere in* $\mathcal{RG}_D$ *for some* $D \in \mathbb{N}$*, then a.s.* G *is recurrent.*

Proposition. *Theorem 2 implies theorem 1.*

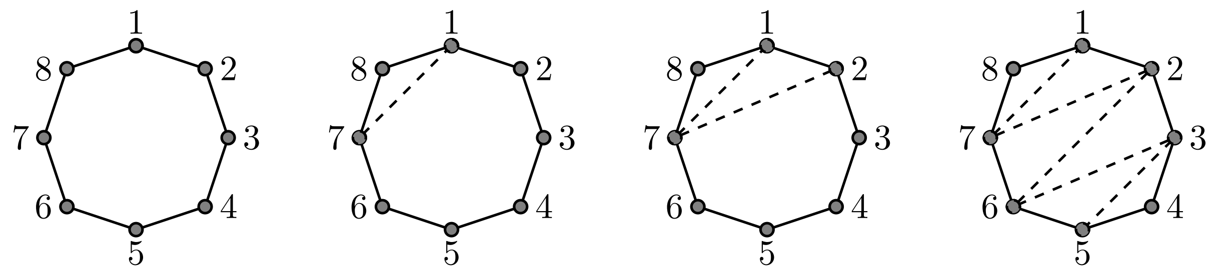
Proof. Let us suppose the proposition is true. We will prove that we can embed any finite planar $G \in \mathcal{RG}_D$ in a triangulation of the sphere T such that T has degree at most $9D$ and $|V(T_i)| \leq 9|V(G_i)|$.

Due to the stereographic projection, we can consider G as a graph on the sphere. For any face of the graph f, we note $v_1, ..., v_n$ the vertices of this face (v_i is connected to v_{i+1} for all $i \in \{1, ..., n-1\}$ and v_n is connected to v_1).

If for all $(i, j) \in \{1, ..., n\}^2$ there is an edge between v_i and v_j iif $|i-j| \equiv 1\ [n]$, then we can triangulate the face by zigzagging: for all $i \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$, we add the vertex (v_i, v_{n-i}) and the vertex (v_i, v_{n-i-1}) (see figure 2).

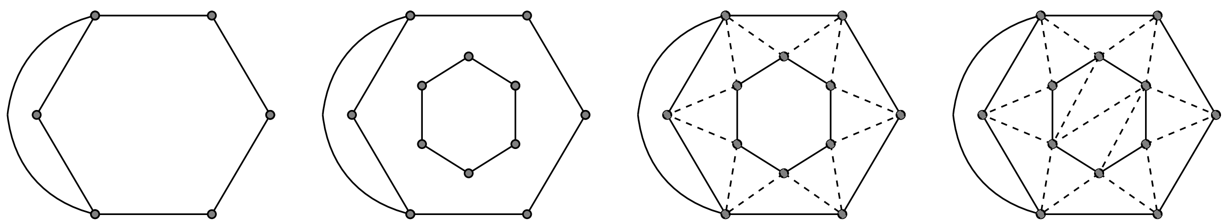
If there are other edges, we add some more vertices inside the face, $u_1, ..., u_{n-1}$. We add the edges linking u_i to u_{i-1}, $u_{(i+1) \bmod (n-1)}$, v_i and v_{i+1}. We can do the first step for the face $u_1, ..., u_n$ (see figure 3).

We can therefore associate a triangulation (T_n, o_n) to each (G_n, o_n). We can take a subsequence of $(T_n, o_n)_{n \in \mathbb{N}}$ that converges to a graph that is a.s. recurrent, and since $\mathbb{P}(o_n \in G_n) \geq \frac{1}{9}$, we can use the Rayleigh monotonicity principle, that states that if a subgraph is transient then the whole graph is transient too, to say that G converges. One can look at [DS06] to learn more on the Rayleigh monotonicity principle. □



(a) 1 (b) 2 (c) 3 (d) 4

Figure 2: Case 1



(a) 1 (b) 3 (c) 3 (d) 4

Figure 3: Case 2

In the next section, we introduce some tools for the proof of the theorem 2. The reader might want to only read the results and skip to 1.2.2 for the proof of the theorem.

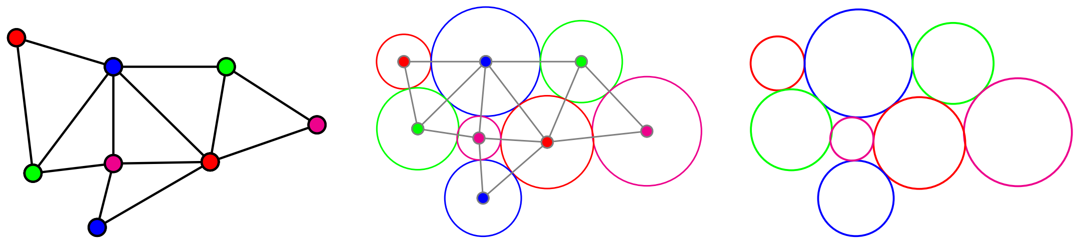
1.2.1 Tools for the Proof

Circle Packings The proof fundamentally depends on the Koebe-Andreev-Thurston's *Circle Packing Theorem.*

Theorem 3. *[Kob36] For any finite planar simply connected triangulation G, there is a circle packing whose tangency graph is G.*

A circle packing is a set of circles on the plane whose intersections are at most punctual. Its tangency graph is the graph whose vertices are the centers of the circles and where two vertices are connected iff the two corresponding circles are tangent.

A proof of theorem 3 is found in [Ste05].



(a) Graph (b) Graph et circles (c) Circle packing

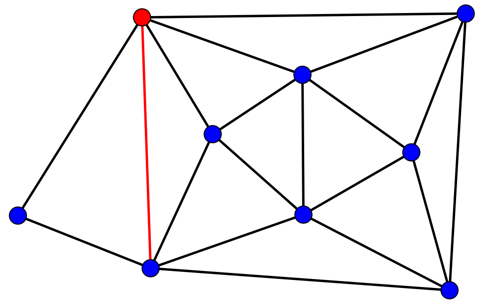
Figure 4: Illustration for the circle packing theorem

Proof. We assume that all triangulations have no vertex that separates the graph in two. If it is the case, we can just apply the proof for the two triangulation we obtained and paste the circle packing together (see case 1 for how to do that). The proof is a bit geometric and the reader might want to refer to the drawings to understand each case of the proof.

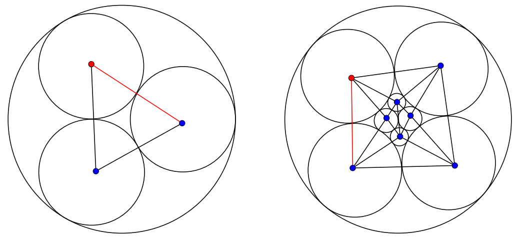
To prove the result, we admit that for any finite triangulation G if we can find a circle packing with at least the right tangencies, that is with eventually more edges than G, then there is a circle packing of G in the unit disc where boundary vertices (a vertex that is in contact with the infinite connected component of $\mathbb{R}^2 \backslash G$) are associated to horocycles (discs tangent to the external disc) and the extraneous tangencies are no more.

We proceed by induction on the number of vertices. If G has 3 vertices, the result is obviously correct. Let us now assume that the result is true for any graph of n vertices or less. For any planar connected triangulation G with $n+1$ vertices, we can take a boundary vertex v. We now have two cases to distinguish.

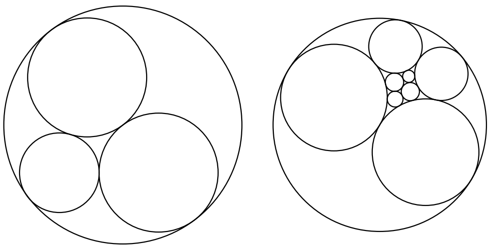
Case 1. If v is connected to another vertex u on the boundary by an interior egde e, then we can cut G in two subgraphs G_1 and G_2 that are separated by e and such that $G_1 \cap G_2 = \{e\}$. By our induction hypothesis and the result we admitted, we have two disc packing in the unit disc P_1 and P_2 for G_1 and G_2, where v and u are associated to horocycles each time. In P_1, the circles P_1^u and P_1^v associated to u and v are tangent and tangent to the unit disc. We can therefore use a Möbius transformation of the disc that sends P_1^u and P_1^v to the circles of radii $1/2$ and centers $(1/2, 0)$ and $(-1/2, 0)$. If we do the same type of tranformation in P_2, we can paste the two circle packings we obtained. We have a circle packing for G.



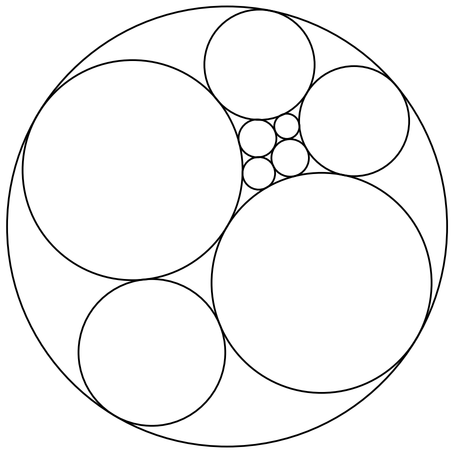
(a) graph



(b) circle packing for the separate graphs



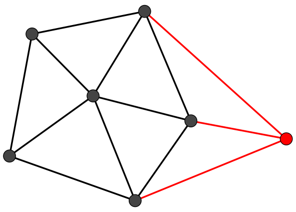
(c) after transformation



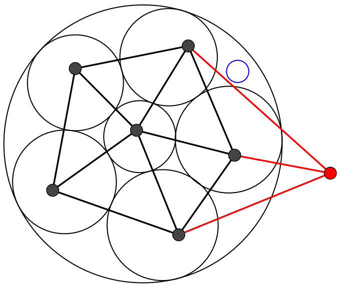
(d) after pasting the packing together

Figure 5: Illustration for case 1

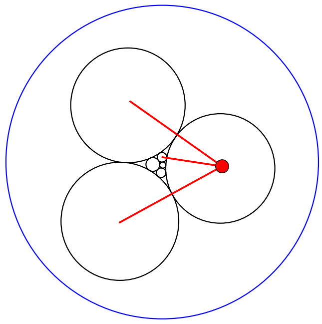
Case 2: If v is only connected to interior vertices, then we can apply induction hypothesis to $G\backslash\{v\}$. We have a circle packing P in the unit disc where the boundary vertices are associated to horocycles. The neighbours of v are therefore horocylces. We consider $\mathbb{R}^2\backslash B(0,1)$ as a disc centered in infinity. We associate this disc to v. We can chose a disc D in the unit disc that is disjoint from the $carr(P)$ and by a Möbius transformation, send D to $\mathbb{R}^2\backslash B(0,1)$. We now have a circle packing for G in the unit disc (with too many edges). With the admitted result, we have the circle packing we wanted.



(a) graph



(b) circle packing for the graph without v



(c) after transformation

Figure 6: Illustration for case 2

In these figure, the red vertex is v and the blue circle is the one on which the inversion is done. In the third figure, we do not have the final circle packing, but we have enough to apply the result we admitted. $\square$

Ring Lemma This result is proven in [RS87].

Theorem 4 (Ring lemma)**.** *If a circle R of radius 1 is surrounded by n circles $R_1, ..., R_n$ of radii $r_1, ..., r_n$ whose tangency graph is a cycle, then there is a $c(n) \in \mathbb{R}$ such that $r_i \geq c(n)$ for all $i \in \{1, ..., n\}$.*

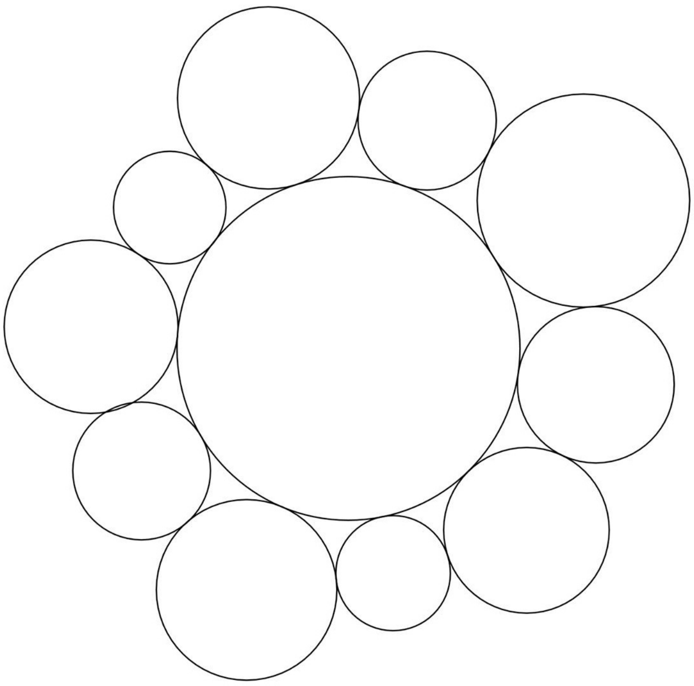


Figure 7: Example of configuration for the Ring lemma

Proof. Let us take R of radius 1 surrounded by n circles. The size of the biggest circle has a uniform lower bound r_m (size of the circle if all the n circles are the same size). The left neighbour of the biggest circle also has a uniform lower bound under which the $n-1$ circles are to remain in the cavity between the biggest circle and R. We can see that in the case where the biggest circle is of radius r_m. If it is bigger, the same lower bound will apply to the radius of the left neighbour.

This operation can be repeated for all the circles which gives us the uniform bound. $\square$

Recurrence/transience The last result we need is a recurrence-transience result on the circle packing. We use a less powerful form of the theorem proven in [BS90].

Definition. *We say that a point p is an accumulation point of a circle packing C if in every neighbourhood of p there are an infinity of circles of C*

Proposition 1. *If a circle packing has no accumulation point, then its tangency graph is recurrent.*

Proof. In order to prove this result, we need to introduce some background. In this section, we only consider circle packings that are **discrete,** that is without points of accumulation, and whose tangency graph are triangulations (which is relevant for what we need). The very important result is the following:

Theorem 5. *A simply connected graph without boundary is the tangency graph of a discrete circle packing on exactly one of the three simply connected Riemann surfaces (the Riemann sphere, the Poincaré disc and the plane).*

This result is proven in [Ste05], we admit it. This is a stronger result than the one we showed before. The circle packing is on the Riemann sphere iff the graph is finite. For the other cases, we have the following theorem:

Theorem 6. *[McC98] Let R be either the disc or the plane and G the tangency graph of a circle packing on R. Then R is the plane iff G is recurrent.*

We prove this result partially. In our case, we know that R is the plane. Therefore, we only need to show that if G is transient, then R is the disc.

We say that a riemannian manifold R is transient if the integral over time of its solution of the heat equation is finite, or equivalently if :

$$\exists U \subset R,\ C > 0,\ \text{such that } \forall \varphi \in \mathcal{C}_c^2(R)\ \left(\int_U \varphi\right)^2 \leq C \int_R \|\nabla\varphi\|^2.$$

We admit the following:

Proposition. *The plane is recurrent and the disc is transient.*

We now need to prove that if the tangency graph of a discrete circle packing is transient, then the surface supporting the circle packing is transient, and is therefore the disc.

In order to do that, we admit that a graph G is transient iff there exists a vertex x_0 and a constant C such that for all $f \in \mathcal{C}(R)$:

$$f(x_0)^2 \leq C \sum_{(x,y)\in E} (f(x) - f(y))^2,$$

where E is the set of edges of G.

These two last results are known as the *Dirichlet criterion.*

We can now prove the proposition. If G, the tangency graph of a discrete circle packing, is transient, let x_0 be a vertex as described in the Dirichlet criterion and $\varphi \in C_c^2(R)$. For all x vertex of G, we consider the union $U(x)$ of all the faces of G that meet x. We take some f such that $f(x)$ is the average of φ over $U(x)$. Therefore we have,

$$f(x_0) = \frac{1}{|U(x_0)|} \int_{U(x_0)} \varphi(x)\ dx \leq \left(C \sum_{(x,y)\in E} (f(x) - f(y))^2 \right)^{\frac{1}{2}}.$$

Furthermore, we have, for any x, y vertices of G:

$$\begin{aligned} f(y) - f(x) &= \frac{1}{|U(x)| \times |U(y)|} \int_{U(x)} \int_{U(y)} \varphi(a) - \varphi(b)\ da\ db \\ &\leq \frac{1}{|U(x)| \times |U(y)|} \int_{V\times V} |\varphi(a) - \varphi(b)|\ da\ db. \end{aligned}$$

Where V is the smallest disc containing $U(x)$ and $U(y)$.

$$\begin{aligned} \int_{V\times V} |\varphi(a) - \varphi(b)|\ da\ db &\leq \int_{V\times V} \int_0^1 ||\nabla\varphi(a + t(b-a))||\ dt\ da\ db \\ &= \int_V ||\nabla\varphi(\nu)||\ |E_\nu|\ d\nu \\ &= \int_V ||\nabla\varphi(\nu)|| \int_V |E_{\nu,a}|\ da\ d\nu \\ &\leq \text{diameter}(V).|V| \int_V ||\nabla\varphi(\nu)||\ d\nu \end{aligned}$$

where $E_\nu = \{(a,b) \in V \times V : \nu \in [a,b]\}$ and $E_{\nu,a} = \{b : \nu \in [a,b]\}$. Since $|V|$, $|U(x)|$ and $|U(y)|$ are all within constant factor by the Ring lemma, we have:

$$(f(x) - f(y))^2 \leq C' \frac{1}{|V|} \left(\int_V ||\nabla\varphi(\nu)||\ d\nu \right)^2 \leq C' \int_V ||\nabla\varphi(\nu)||^2\ d\nu$$

for some constant C. This gives us:

$$\left(\int_{U(x_0)} \varphi(x) dx \right)^2 = f(x_0)^2 \leq \sum_{(x,y)\in E} (f(x) - f(y))^2 \leq K \int_R ||\nabla\varphi(\nu)||^2\ d\nu$$

The right hand inequality is another consequence of the Ring lemma and the summation over all edges. Therefore R is transient, it is the disc. □

1.2.2 Recurrence for triangulations

Theorem. *Let (G, o) be the Benjamini-Schramm limit of a sequence $(G_n)_{n\in\mathbb{N}}$ of unbiased, finite triangulations of the sphere in $\mathcal{RG}_D$ for some $D \in \mathbb{N}$, then a.s. G is recurrent.*

Proof. For all $n \in \mathbb{N}$, we can use the Circle Packing Theorem to associate to G_n a random circle packing P^n. We note P_v^n the circle associated to the vertex v in P^n. We can consider that $P_{o_n}^n$ is $B(0, 1)$.

There are only three circles in P^n that are in contact with the infinite component of $\mathbb{R}^2 \backslash P^n$. These are the three vertices of the triangle containing the opposite pole from which we do the stereographic projection. We note t_1, t_2, t_3 the associated circles.

Any circle of $P^n \backslash \{t_1, t_2, t_3\}$ is surrounded by other circles, we can therefore use the Ring Lemma.

We take $k \in \mathbb{N}^*$. Since $|V(G_n)| \to \infty$, with high probability, $B_{G_n}(o_n, k) \cap \{t_1, t_2, t_3\} = \varnothing$, $B_{G_n}(o_n, k)$ is the ball of center o_n and radius k for the combinatoric distance. By the Ring Lemma, there is a constant c depending only on D and k such that for all circle at a combinatoric distance less than k from o_n, the radius is in $[1/c, c]$.

By compactness, there is a subsequence of P^n that converges to a random circle packing P whose tangency graph is G.

Proposition 2. *There is at most one accumulation point in P.*

Proof. To prove this proposition, a lemma is needed.

Definition. *Let C be a finite set of points in $\mathbb{R}^2$. We define the isolation radius of a point $w \in C$ as $\rho_w = min\{|v - w|, v \in C \backslash \{w\}\}$.*

We say that w is (δ, s)-supported ($\delta \in]0, 1[$, $s \in \mathbb{N}$, $s \geq 2$) if the circle $B(w, \delta^{-1}\rho_w)$ contains at least s points of C that are not in $B(p, \delta\rho_w)$ for any $p \in C$.

Lemma 1. *The number of (δ, s)-supported points in C is at most $\frac{K|C|}{s}$ for some constant K that depends only on δ.*

We will prove this lemma at the end of this section. If it is true, and if there is a positive probability α that there are two accumulation points p_1, p_2 in C, then we can take δ such that $p_1, p_2 \in B(0, \delta^{-1})$ and such that $|p_1 - p_2| \geq 3\delta$. Then for arbitrarily large s, there are infinitely many j such that with probability α, o_j is (δ, s)-supported in the set of centers of P^j. Since o_j is chosen uniformly, it means that a proportion α of the centers are (δ, s)-supported for arbitrarily large s. This contradicts the lemma.

□

If P has no accumulation point, by the result we showed earlier, G is recurrent.

If P has one accumulation point p, we consider the graph F whose vertices are the center of the circle at a geometric distance less than one to p, the edges being the same as in G. $G \backslash F$ has no accumulation point, thus it is recurrent. Moreover, by inverting the circle of center p and radius 1, we obtain a graph isomorphic to F with non accumulation point, thus recurrent. Since $G \backslash F$ and F are connected by a finite number of edges, G is recurrent. □

1.2.3 Proof of the lemma

Lemma. *The number of (δ, s)-supported points in C is at most $\frac{K|C|}{s}$ for some constant K that depends only on δ.*

Proof. In order to prove the lemma, we need to work on square tiling of the plane.

Let $k \in \mathbb{N}$ such that $k \geq 3$. We consider a sequence $(\Gamma_n)_{n\in\mathbb{Z}}$ of square tiling, where for all $n \in \mathbb{Z}$, all the square in Γ are the same size and each is tiled by k^2 squares of Γ_{n-1}.

We can assume that no point of C is on the boundary of any square in $\bigcup_{n\in\mathbb{Z}} \Gamma_n$.

Let $n \in \mathbb{Z}$, we say that $S \in \Gamma_n$ is s-supported if for every $S' \in \Gamma_{n-1}$ we have $|C \cap S \backslash S'| \geq s$ $(s \geq 2)$. To estimate the number σ of s-supported squares in $\bigcup_{n\in\mathbb{Z}} \Gamma_n$, we introduce the flow f:

$$f(S', S) := \begin{cases} min(s/2, |S' \cap C|) & if\ S \in \Gamma_n,\ S' \in \Gamma_{n-1},\ S' \subset S \\ 0 & if\ S \in \Gamma_n,\ S' \in \Gamma_{n-1},\ S' \not\subseteq S \\ -f(S, S') & if\, S' \in \Gamma_n,\ S \in \Gamma_{n-1} \\ 0 & \text{otherwise} \end{cases}$$

We take $a \in \mathbb{Z}$ small enoug such that any square of Γ_a contains at most one point of C. Then, we have:

$$\sum_{S' \in \Gamma_a} \sum_{S \in \Gamma_{a+1}} f(S', S) = |C|$$

And for any $b \in \mathbb{Z} \cap]a; +\infty[$, we have:

$$\sum_{S' \in \Gamma_b} \sum_{S \in \Gamma_{b+1}} f(S', S) \geq 0$$

Therefore:

$$\sum_{n=a+1}^{b} \sum_{S \in \Gamma_a} \sum_{S' \in \cup_{n\in\mathbb{Z}} \Gamma_n} f(S', S) \leq |C|$$

We know by definition that for any $S \in \bigcup_{n\in\mathbb{Z}} \Gamma_n$, $\sum_{S' \in \cup_{n\in\mathbb{Z}}} f(S', S) \geq 0$, and if S is s-supported, then $\sum_{S' \in \cup_{n\in\mathbb{Z}}} f(S', S) \geq s/2$. We have:

$$\sigma \frac{s}{2} \leq \sum_{n=a+1}^{b} \sum_{S \in \Gamma_a} \sum_{S' \in\ \cup_{n\in\mathbb{Z}} \Gamma_n} f(S', S) \leq |C|$$

Therefore $\sigma \leq |C| \frac{2}{s}$.

We can now estimate the number of (δ, s)-supported points in C with σ. We now choose $\Gamma = (\Gamma_n)_{n\in\mathbb{Z}}$ randomly. We take $k = \lfloor 20\delta^{-1} \rfloor$. We want to have a distribution of Γ such that $\cup_{n\in\mathbb{Z}} \Gamma_n$ has a distribution invariant under rescaling and translation and such that the Γ_0 has a radius in $[1; k[$.

We take a random sequence $(\alpha_n)_{n\in\mathbb{Z}}$ where the α_n are independent and uniformly distributed in $\{1, ..., k-1\}^2$ and β independent from $(\alpha_n)_{n\in\mathbb{Z}}$ and chosen uniformly in $[0; \log(k)[$ and we define:

$$\Gamma_n = \{exp(\beta) k^n [j; j+1] \times [j'; j'+1] + exp(\beta) \sum_{m=-\infty}^{n-1} k^m \alpha_m,\ j, j' \in \mathbb{Z}\}$$

We say that point $w \in C$ is a **city** in a square $S \in \bigcup_{n\in\mathbb{Z}} \Gamma_n$ if the size of S are of length in $[4\rho_w; 5\rho_w]$ and the distance between w and the center of S is at most $\delta^{-1} \rho_w$.

If w is a city of a square S and w is (δ, s)-supported in C, then S is s-supported. Moreover, there is a constant c_0 that depends only on k such that a point in C has probability at least c_0 of being a city for a certain square of$S$. If we note N the number of (δ, s)-supported points in C, and γ the number of couple (w, S) where w is a city of S and S is s-supported, the $c_0 N \leq \gamma$. Moreover, there is a constant c_1 such that a square cannot have more that c_1 cities in it (by area considerations). We have:

$$Nc_0/c_1 \leq \sigma \leq |C|\frac{2}{s}$$

The lemma is proven. $\square$

2 Study of a Group Walk Random Graph

A Group Walk Random Graph F is defined as follow. Take G a locally finite, that is with vertices of finite degree, graph. Take R a recurrent set of vertices of G, this is the deterministic set of vertices of our random graph F.

From each vertex x of R, we start (independently from each vertex) a simple random walk on G. Let $y \in R$ be the first vertex in R that the random walk meets, then we add the vertex (x, y) to the set of vertex of F.

We reiterate this operation independently a random number (following a law $\mathcal{P}$) of times independently for each vertex of R and the random graph is constructed.

The graph we obtain is not necessarily simple, there can be more than one edge between two vertices of F. However, this does not impact our study and the reader might not want to consider the extra edges.

In this paper, see fig. 8, G is the canopy tree, R is the set of the leaves and $\mathcal{P}$ follows a Poisson distribution of parameter λ. To have more details on GWRG, one can refer to [Geo15].

We count the height with the leaves being of height 0 and the parents being of height 1 more than their children. We note $h(x)$ the height of a vertex x. We call the vertex that is on the left hand of all others vertices the root.

We define the confluent of two leaves as the height of their smallest common ancestor.

We also define L_h to be the set of vertices that have a confluent with the root equal to h.

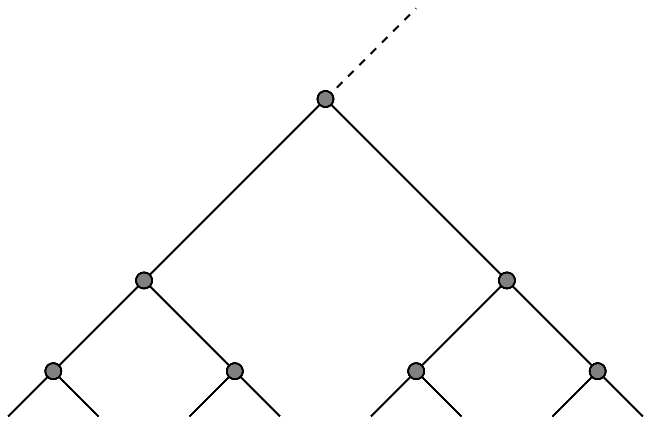


Figure 8: Canopy tree, where L_0 to L_3 are pictured

Proposition 3. *Let $x, y \in R$, the probability that a random walk from x ends in y is $\Theta(4^{-k})$ where k is the height of the confluent between x and y.*

Proof. Let us note H the maximum height of the random walk. If $(X_i)_{i\in\mathbb{N}}$ is our random walk from x, then $2^{h(X_i)}$ is a martingale and by the optional stopping theorem we have:

$$\mathbb{P}(H \geq h) = \frac{1}{2^h - 1}$$

Therefore, we have:

$$\mathbb{P}(H = h) = \frac{2^h}{(2^h - 1)(2^{h+1} - 1)}$$

We can now calculate

$$\begin{aligned}\mathbb{P}(X_\infty = y) &= \sum_{h=k}^{\infty} \mathbb{P}(x \to y | H = h)\mathbb{P}(H = h) \\ &= \sum_{h=k}^{\infty} \frac{1}{2^h} \frac{2^h}{(2^h - 1)(2^{h+1} - 1)}\end{aligned}$$

We have $\mathbb{P}(X_\infty = y) = \Theta(4^{-k})$ □

It is easy to see that the number of edges between x and y follows a Poisson law of parameter $2\lambda\mathbb{P}(X_\infty = y)$. In the rest of the paper, to light notations, we will consider that $\mathbb{P}(X_\infty = y) = 4^{-k}$. This simplification changes the parameters we consider by at most a constant, so all results also hold for the original model.

2.1 Distribution of the simple degree

The simple degree of the graph is the number of vertices that send at least one walk to the root. The calculus its mean has been done in [GH16].

2.1.1 The mean

Proposition 4. *The mean of the simple degree is $\Theta(\sqrt{\lambda})$.*

Proof. The simple degree can be described as $\delta = \sum_{h \in \mathbb{N}} \sum_{x \in L_h} \delta_x$ where δ_x are independent Bernoulli's Law of parameter $(1 - \exp^{-\lambda 4^{-h}})$, h being the smallest k such that $x \in L_k$.

Therefore we have:

$$\begin{aligned}\mathbb{E}[\delta] &= \sum_{h \in \mathbb{N}} \sum_{x \in L_h} \mathbb{E}[\delta_x] \\ &= \sum_{h \in \mathbb{N}} \sum_{x \in L_h} (1 - \exp^{-\lambda 4^{-h}}) \\ &= \sum_{h \in \mathbb{N}} 2^h (1 - \exp^{-\lambda 4^{-h}})\end{aligned}$$

We take $N = \log_4(\lambda)$,

$$\mathbb{E}[\delta] = \underbrace{\sum_{h=0}^{N} 2^h (1 - \exp^{-\lambda 4^{-h}})}_{A} + \underbrace{\sum_{h=N+1}^{\infty} 2^h (1 - \exp^{-\lambda 4^{-h}})}_{B}$$

For $h \leq N$, $(1 - exp(-1)) \leq (1 - \exp^{-\lambda 4^{-h}}) \leq 1$, thus $A = \Theta(2^N) = \Theta(\sqrt{\lambda})$.
For $h \gg N$, $(1 - \exp^{-\lambda 4^{-h}}) = \Theta(\lambda 4^{-h})$, we have:

$$B = \Theta(\sum_{h \geq N} 2^h \lambda 4^{-h}) = \Theta(\frac{\lambda}{\sqrt{\lambda}})$$

□

2.1.2 The variance

Proposition 5. *The variance of the simple degree is $\Theta(\sqrt{\lambda})$.*

Proof. We use the same notations as before to describe the simple degree.
We have

$$\begin{aligned}
\mathbb{E}[\delta^2] &= \mathbb{E}\left[\left(\sum_{h\in\mathbb{N}}\sum_{x\in L_h}\delta_x\right)^2\right] &(1)\\
&= \mathbb{E}\left[\sum_{h\in\mathbb{N}}\sum_{x\in L_h}\delta_x^2\right] + \mathbb{E}\left[\sum_{h\in\mathbb{N}}\sum_{x\in L_h}\left(\delta_x\sum_{k\in\mathbb{N}}\sum_{y\in L_k,y\neq x}\delta_y\right)\right] &(2)\\
&= \sum_{h\in\mathbb{N}}\sum_{x\in L_h}\mathbb{E}\left[\delta_x^2\right] + \sum_{h\in\mathbb{N}}\sum_{x\in L_h}\left(\mathbb{E}\left[\delta_x\right]\sum_{k\in\mathbb{N}}\sum_{y\in L_k,y\neq x}\mathbb{E}\left[\delta_y\right]\right) &(3)\\
&= \sum_{h\in\mathbb{N}}\sum_{x\in L_h}\mathbb{E}\left[\delta_x^2\right] + \sum_{h\in\mathbb{N}}\sum_{x\in L_h}\left(\mathbb{E}\left[\delta_x\right]\sum_{k\in\mathbb{N}}\sum_{y\in L_k}\mathbb{E}\left[\delta_y\right]\right) - \sum_{h\in\mathbb{N}}\sum_{x\in L_h}\mathbb{E}\left[\delta_x\right]^2 &(4)\\
&= \underbrace{\sum_{h\in\mathbb{N}}\sum_{x\in L_h}(\mathbb{E}\left[\delta_x^2\right] - \mathbb{E}\left[\delta_x\right]^2)}_{=Var(\delta)} + \underbrace{\sum_{h\in\mathbb{N}}\sum_{x\in L_h}\mathbb{E}\left[\delta_x\right]\sum_{k\in\mathbb{N}}\sum_{y\in L_k}\mathbb{E}\left[\delta_y\right]}_{=\mathbb{E}[\delta]^2} &(5)
\end{aligned}$$

To get from (3) to (4) we use the independence of the laws. Since we know that $\mathbb{E}[\delta] = \Theta(\sqrt{\lambda})$, we only have to calculate $Var(\delta)$.

$$\begin{aligned}
Var(\delta) &= \sum_{h\in\mathbb{N}}\sum_{x\in L_h}(\mathbb{E}\left[\delta_x^2\right] - \mathbb{E}\left[\delta_x\right]^2)\\
&= \sum_{h\in\mathbb{N}}\sum_{x\in L_h}(\mathbb{E}\left[\delta_x\right] - \mathbb{E}\left[\delta_x\right]^2)\\
&= \sum_{h\in\mathbb{N}}2^h((1-\exp^{-\lambda 4^{-h}}) - (1-\exp^{-\lambda 4^{-h}})^2)\\
&= \sum_{h\in\mathbb{N}}2^h((1-\exp^{-\lambda 4^{-h}})\exp^{-\lambda 4^{-h}})
\end{aligned}$$

We take $N = log_4(\lambda)$.

For $h \leq N$, we have $(1-\exp^{-\lambda 4^{-h}})\exp^{-\lambda 4^{-h}} = \Theta(1)$

$$\sum_{h=0}^{N}2^h((1-\exp^{-\lambda 4^{-h}})\exp^{-\lambda 4^{-h}}) = \Theta(2^N) = \Theta(\sqrt{\lambda})$$

For $h \gg N$, we have $(1-\exp^{-\lambda 4^{-h}})\exp^{-\lambda 4^{-h}} = \Theta(\lambda 4^{-h}(1-\lambda 4^{-h}))$

Therefore

$$\sum_{h=N}^{\infty} 2^h(1-\exp^{-\lambda 4^{-h}})\exp^{-\lambda 4^{-h}} = \Theta(\sum_{h=N}^{\infty} 2^h\lambda 4^{-h}(1-\lambda 4^{-h})) \tag{6}$$

$$= \Theta(\sum_{h=N}^{\infty} \lambda 2^{-h} - \lambda^2 8^{-h})) \tag{7}$$

$$= \Theta(2\lambda \underbrace{2^{-N}}_{\sqrt{\lambda}} - \frac{8}{7}\lambda^2 \underbrace{8^{-N}}_{=\lambda\sqrt{\lambda}})) \tag{8}$$

$$= \Theta(\sqrt{\lambda}) \tag{9}$$

We have $Var(\delta) = \Theta(\sqrt{\lambda})$. $\square$

2.1.3 The distribution

We note the simple degree δ. In this section we will calculate $\mathbb{P}(\delta = k)$ for any $k \in \mathbb{N}$. We note p_x the probability that there is an edge from x to the root. Thus $p_x = (1-\exp(-\lambda 4^{-h}))$ where h is the height of the confluent between the root and x.

Proposition 6. *Let $k \in \mathbb{N}$, we have $\mathbb{P}(\delta = k) = \Theta(\exp(-2\lambda)\sqrt{\lambda}^k)$ when $\lambda \to \infty$.*

Proof. Let us calculate $\mathbb{P}(\delta = 0)$.

$$\begin{aligned}
\mathbb{P}(\delta = 0) &= \prod_{h=0}^{\infty}\prod_{x\in L_h}(1-p_x)\\
&= \prod_{h=0}^{\infty}\prod_{x\in L_h}\exp(-\lambda 4^{-h})\\
&= \prod_{h=0}^{\infty}\exp(-\lambda 2^{-h})\\
&= \exp(-\lambda\sum_{h=0}^{\infty}2^{-h})\\
&= \exp(-2\lambda)
\end{aligned}$$

Let us take $k \in \mathbb{N}^\star$. We first prove that $\mathbb{P}(\delta = k) = O(\exp(-2\lambda)\sqrt{\lambda}^k)$.

$$\begin{aligned}
\mathbb{P}(\delta = k) &= \sum_{\substack{x_1,\dots,x_k \in L_\infty \\ x_i \neq x_j}} \prod_{i=1}^{k} p_{x_i} \prod_{x \notin \{x_1,\dots,x_k\}} (1 - p_x) \\
&= \sum_{\substack{x_1,\dots,x_k \in L_\infty \\ x_i \neq x_j}} \prod_{i=1}^{k} \frac{p_{x_i}}{1 - p_{x_i}} \mathbb{P}(\delta = 0) \\
&\leq \mathbb{P}(\delta = 0) \sum_{x_1,\dots,x_k \in L_\infty} \prod_{i=1}^{k} \frac{p_{x_i}}{1 - p_{x_i}} \\
&= \mathbb{P}(\delta = 0) \left(\sum_{x \in L_\infty} \frac{p_x}{1 - p_x} \right)^k \\
&= \mathbb{P}(\delta = 0) \left(\sum_{h=0}^{\infty} \sum_{x \in L_h} \exp(\lambda 4^{-h}) - 1 \right)^k \\
&= \Theta(\exp(-2\lambda)\sqrt{\lambda}^k)
\end{aligned}$$

The last calculus is the same as the one we did in the calculus of the mean.

In order to prove $\exp(-2\lambda)\sqrt{\lambda}^k = O(\mathbb{P}(\delta = k))$, we introduce the following.

Definition. *Let h such that $2^h \geq k$ and $(L^k_{h,i})_{i \in \{1,\dots,k\}}$ be a partition of L_h such that $\#L^k_{h,i} = \lfloor \frac{2^h}{k} \rfloor + 1$ for $i \leq k - 1$. $L^k_{h,k}$ is defined as the remaining, thus $\#L^k_{h,k} \leq \lfloor \frac{2^h}{k} \rfloor$*

The idea here is to have a product of k sums that do not depend on each other. In order to do that, we only take into account a different part of the vertices in each sum. This gives us a lower bound for $\mathbb{P}(\delta = k)$.

$$\begin{aligned}
\mathbb{P}(\delta = k) &= \sum_{h_1=0}^{\infty} \sum_{x_1 \in L_{h_1}} \dots \sum_{h_k=0}^{\infty} \sum_{\substack{x_k \in L_{h_k} \\ x_k \notin \{x_1,\dots,x_{k-1}\}}} \prod_{i=1}^{k} p_{x_i} \prod_{x \notin \{x_1,\dots,x_k\}} (1 - p_x) && (10) \\
&= \sum_{h_1=0}^{\infty} \sum_{x_1 \in L_{h_1}} \dots \sum_{h_k=0}^{\infty} \sum_{\substack{x_k \in L_{h_k} \\ x_k \notin \{x_1,\dots,x_{k-1}\}}} \prod_{i=1}^{k} \frac{p_{x_i}}{1 - p_{x_i}} \mathbb{P}(\delta = 0) && (11) \\
&\geq \mathbb{P}(\delta = 0) \sum_{h_1=0}^{\infty} \sum_{x_1 \in L^k_{h_1,1}} \dots \sum_{h_k=0}^{\infty} \sum_{x_k \in L^k_{h_k,k}} \prod_{i=1}^{k} \frac{p_{x_i}}{1 - p_{x_i}} && (12) \\
&= \mathbb{P}(\delta = 0) \sum_{h_1=0}^{\infty} \sum_{x_1 \in L^k_{h_1,1}} \frac{p_{x_1}}{1 - p_{x_1}} \dots \sum_{h_k=0}^{\infty} \sum_{x_k \in L^k_{h_k,k}} \frac{p_{x_k}}{1 - p_{x_k}} && (13) \\
&\geq \mathbb{P}(\delta = 0) \left(\frac{1}{2k} \right)^k \left(\sum_{h=0}^{\infty} \sum_{x \in L_h} \exp(-\lambda 4^{-h}) - 1 \right)^k && (14) \\
&= \Theta(\exp(-2\lambda)\sqrt{\lambda}^k) && (15)
\end{aligned}$$

We can go from (13) to (14) because
$\sum_{h_i=0}^{\infty} \sum_{x_i \in L_{h,1}^k} \frac{p_{x_1}}{1-p_{x_1}} \geq \frac{1}{2k} \sum_{h=0}^{\infty} \sum_{x \in L_h} \exp(-\lambda 4^{-h}) - 1$.

Moreover the calculus of $\sum_{h=0}^{\infty} \sum_{x \in L_h} \exp(-\lambda 4^{-h}) - 1$ is the same as the one that has been done in the previous sections.

□

2.2 Size of the largest clique containing the root

Here we calculate the size $S \in \mathbb{N}$ of the largest complete subgraph containing the root. We do not take into account the edges that originate from the root and we actually have an estimation of S when the subgraph contains all the first vertices.

Proposition 7. *There are two deterministc constant c and c' such that, with probability going to one as λ goes to infinity, $c\sqrt{\frac{\lambda}{\log(\lambda)}} \leq S \leq c'\sqrt{\frac{\lambda}{\log(\lambda)}}$.*

Proof. In order to have that estimation, we only need to compute the probability that the first 2^n vertices are in the graph. We note p_n that probability.

$$\begin{aligned} p_n &= p_{n-1}^2 (1 - \exp(-\lambda 4^{-n}))^{2^n} \\ &= \prod_{i=1}^{n} (1 - \exp(-\lambda 4^{-i}))^{4^{n-i} 2^i} \end{aligned}$$

Thus we have, $p_n \leq (1 - \exp(-\lambda 4^{-n}))^{2^n}$. We take $\lambda = \frac{n}{2} 4^n$, we have $p_n \leq (1 - \exp(-\frac{n}{2}))^{2^n}$.
We want to prove that $p_n \to 0$, which is the same as $\log(p_n) \to -\infty$.
We have $\log(p_n) \leq 2^n \log(1 - \exp(-\frac{n}{2})^)$.
We also have $\log(1 - \exp(-\lambda 4^{-n})) \sim \exp(-\frac{n}{2})$ and $2^n \exp(-\frac{n}{2}) \to -\infty$, we have the result.

We can also use the Bernoulli's inequality.

$$\begin{aligned} p_n &\geq \prod_{i=1}^{n} (1 - \exp(-\lambda 4^{-i}) 4^{n-i} 2^i) \\ &\geq \prod_{i=1}^{n} (1 - \exp(-\lambda 4^{n}) 4^{n}) \end{aligned}$$

We take $\lambda = 4n \log(2) 4^n$, we have $p_n \geq 1 - \exp(-4n \log(2)) 4^n \to 1$

We have the result, this is equivalent to say that we have a clique of size $\Theta(\sqrt{\frac{\lambda}{\log(\lambda)}})$. We now want to prove that there is clique bigger by more than a constant factor than this one.

There is two steps to this process, first we prove that for N greater than $\sqrt{\lambda log(\lambda)}$ the root is not link to more vertices than $\Theta(\sqrt{\frac{\lambda}{\log(\lambda)}})$ further away than $N 2^N$. Then we prove that if a clique as more than $\Theta(\sqrt{\frac{\lambda}{\log(\lambda)}})$ vertex in one tree, it is not likely to expand. Therefore the root is in a clique of size at most $\Theta(\sqrt{\frac{\lambda}{\log(\lambda)}})$.

We can easily calculate the average number X of neighbours of the root which have a confluent greater than $\log_4(\lambda log(\lambda)))$ with the root. It is the same sum as in the calculus of the mean of the simple degree, from a different starting point.

$$\mathbb{E}[X] = \sum_{h=N+1}^{\infty} 2^h (1 - \exp^{-\lambda 4^{-h}})$$

With $N = \lfloor \log_4(\lambda log(\lambda))) \rfloor$. With the same type of argument as before, we have:

$$\begin{aligned}
\mathbb{E}[X] &= \Theta\left(\sum_{h=N+1}^{\infty} 2^h \lambda 4^{-h}\right) \\
&= \Theta(\lambda 2^{-N}) \\
&= \Theta\left(\sqrt{\frac{\lambda}{\log(\lambda)}}\right)
\end{aligned}$$

Therefore, if there is another clique than the one we already know, it has to be in the first 2^N vertices.

We want to divide these 2^N vertices into trees of the same size. We take $n \in \mathbb{N}$ such that $2^N = n2^n$. We have $2^n = \Theta(\sqrt{\frac{\lambda}{\log(\lambda)}})$.

Let us take $\alpha \in)0; 1($. If there is a clique of size $\Theta(\sqrt{\frac{\lambda}{\log(\lambda)}})$, as shown in the next paragraph, with high probability, one of those trees $\mathbb{T}$ contains $\alpha\sqrt{\frac{\lambda}{\log(\lambda)}}$ vertices of that clique. The probability that a vertex from a different tree is linked to $\alpha\sqrt{\frac{\lambda}{\log(\lambda)}}$ vertices $\mathbb{T}$ is lower than $p = 2^n(1 - \exp(-4^{-n}\lambda))^{\alpha\sqrt{\frac{\lambda}{\log(\lambda)}}}\sqrt{\frac{\lambda}{\log(\lambda)}}$. Moreover, the probability that k vertices are linked to $\alpha\sqrt{\frac{\lambda}{\log(\lambda)}}$ vertices $\mathbb{T}$ is lower than p^k. Therefore, the probability that there is a clique in those two trees is lower than $\frac{p}{1-p}$. Summing on all the trees, we have $\frac{np}{1-p}$, this goes to 0 when λ goes to infinity. Therefore, there is a very little chance that a clique bigger than the one we know and this probability goes to zero.

If none of the subtrees contains $\alpha\sqrt{\frac{\lambda}{\log(\lambda)}}$ vertices of that clique, then a constant proportion of edges go between subtrees. The probability of a specific set of vertices of that form being a clique is at most $(1 - \exp(-4^n\lambda))^{c\frac{\lambda}{\log(\lambda)}}$ for $c = \frac{1-\alpha}{2}$. But the number of possible subsets is at most $(2^N)^{\sqrt{\frac{\lambda}{\log(\lambda)}}}$, so the probability that any of them work is at most $(2^N)^{\sqrt{\frac{\lambda}{\log(\lambda)}}}(1 - \exp(-4^n\lambda))^{(c\frac{\lambda}{\log(\lambda)}))}$, which goes to zero.

We have the result. $\square$

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