Distortion Colourings of (3,4)-Biregular Graphs

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1 Introduction

A popular problem in combinatorics is the proper edge colouring problem for a given graph. I.e. what is the least number of colours required to colour the edges of that given graph without two incident edges having the same colour? A specific variation of this problem is called the proper distortion colouring problem for a bipartite graph using n colours, say. The problem is posed as follows:

Let the bipartition of the graph be {A,B}. For each edge e $\in$ G let there be a fixed permutation σ_e associated to that edge which permutes the n colours. We now let an edge have 2 colours, one for each side, like so:

Suppose that the A side of the edge e is coloured x $\in$ {0,1,2,...,n-1}. Then the B side of that edge will be coloured σ_e(x). The permutation essentially 'distorts' the colour of the edge, hence the name distortion colouring. Note that the set of possible permutations that can be associated to an edge is dependent on the number of colours used. Namely if n colours are used then each permutation is essentially an element of S_n.

The aim of this short paper is to show that there is a proper distortion colouring for a specific type of bipartite graph using 5 colours. The graph in question is a specific type of biregular graph.

Definition 1. *An (s,t)-biregular graph G is a bipartite graph,i.e $G = (A \cup B, E)$ such that each vertex in A has degree s, and each vertex in B has degree t.*

Definition 2. *A proper edge distortion colouring for a bipartite graph G (with respect to the permutations on the edges) is an assignment of colours to both sides of all the edges such that:*

1) If an edge $e \in G$ has its A side coloured x then its B side will automatically be coloured $\sigma_e(x)$.

2) For any $v \in V(G)$ the edges incident to v all have different colours when 'seen' from the vertex v.

In this paper we shall prove that an arbitrary (3,4) biregular multigraph (with a finite number of vertices) has a proper distortion colouring using 5 colours regardless of the permutations on the edges. This is thus an extension of the result proven in [1]. In that paper it is proven that an arbitrary finite 3 regular bipartite graph has a proper distortion colouring using 4 colours regardless of the permutations on the edges. The concepts in that paper will be referred to quite often here and thus it is advised that the reader reads that paper before continuing with this one.

2 Main

Theorem 2.1. *For every $(3,4)$-biregular multigraph G, and any edge distortions, there is a proper distortion-colouring of E(G) using 5 colours 0,1,2,3,4.* .

Proof. Using an edge counting argument we see that there are 3|A| edges going out of A and 4|B| edges going into B. Since G is bipartite that implies that 3|A| = 4|B|. Thus $\frac{3}{4}$ |A| = |B|. Now |B| $\in \mathbb{N}$ so that means |A| = 4m for some m $\in \mathbb{N}$ and |B| = 3m.

Now we shall artificially add m vertices to the partite set B. Let each vertex in the set of extra vertices (this set of vertices will be denoted P from now on), have degree 4 such that the edges going between P and A cover each vertex in A exactly once. From now on any edge going between P and A will be called a 'fake' edge, and in diagrams a fake edge will be coloured green. 'Real' edges will usually be coloured black, except for later on in the proof where they may also be coloured blue. This will be explained when the need arises. See the example graph below for illustrative purposes.

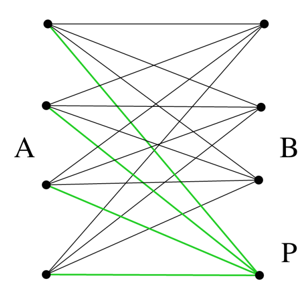


Figure 1: A 4-regular bipartite graph with fake edges going between vertex sets A and P.

Let $\tilde{G}$ denote the graph obtained from G by adding the vertices and fake edges as described above. Now since $\tilde{G}$ is 4-regular that means that there exists a perfect matching M in $\tilde{G}$ [2, Corollary 2.1.3]. Now let G':= $\tilde{G}$- M, i.e. G' is the graph obtained from $\tilde{G}$ by removing the matching M. G' is 3 regular and thus has a proper distortion colouring using 4 colours [1, Theorem 3.1] regardless of the permutations on the edges. As it turns out the same result holds when using 5 colours instead of 4, namely:

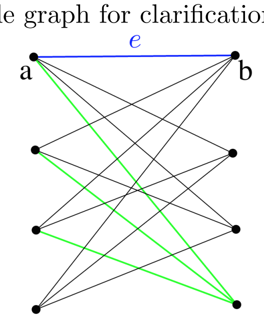
Corollary 2.1.1. *For every 3 regular bipartite multigraph G, and any edge distortions, there is a proper distortion-colouring of E(G) using 5 colours 0,1,2,3,4.*

Please refer to the appendix for the proof of this corollary. The proof is almost identical to the proof in [1, Theorem 3.1] with a few minor adjustments for the fact that 5 colours are used instead of 4.

Now to prove theorem 2.1, firstly observe that when removing the matching M from $\tilde{G}$ to obtain G' we remove m false edges and 3m 'real' edges. (i.e. those edges that also exist in G)

The idea is to add all of the 3m **real** edges in M back to G' whilst preserving a fixed proper edge distortion colouring for G' which we know exists via corollary 2.1.1. Then we can simply remove the remaining fake edges and dummy vertices (vertices in P) which would then yield that there is proper edge distortion colouring for G using 5 colours no matter what permutations are on the edges, as required.

Let us begin by attempting to add a single real edge e ∈ M back to G'. From now on real edges from M that are added to G' will be coloured blue instead of black. See below an example graph for clarification.



Let the endpoints of e be vertices a ∈ A, b ∈ B, i.e. e = ab.

Firstly observe that there are 3 real edges (not including e) incident to the vertex b ∈ B which implies that there are 2 more distinct colours 'available' for the B side of the edge e. Let x,y ∈{01,2,3,4} be the those available colours. Let σ_e^{-1}(x) and σ_e^{-1}(y) be the corresponding colours required for the A side of the edge e.

Case 1: At least 1 of the colours σ_e^{-1}(x) and σ_e^{-1}(y) is available at the vertex a ∈ A. Then we are done. Just use one of those colours for the A side of the edge e.

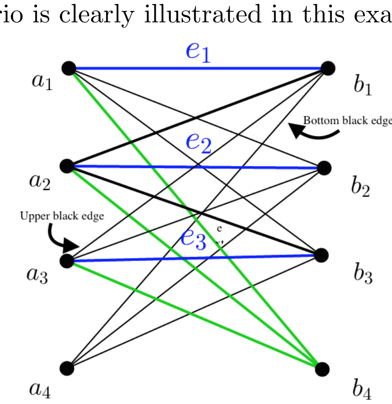
Case 2: Neither of the 2 colours are available, i.e. the real edges incident to the vertex a ∈ A have the colours σ_e^{-1}(x) and σ_e^{-1}(y) on their A side.

Without loss of generality consider the edge incident to vertex a ∈ A coloured σ_e^{-1}(x) on its A side. Now look at its B side. There are 2 more colours available for its B side. At least one of which will correspond to neither σ_e^{-1}(x) nor σ_e^{-1}(y) on the A side of the edge. Use that colour for its B side. However what if the corresponding colour for that edge at a ∈ A 'clashes' with the colour of the fake edge incident to a ∈ A?

Well, firstly observe that we need to fix the colour on the B side of that fake edge with the colour assigned to it by corollary 2.1.1. This is to avoid disrupting the proper distortion colouring on G' that we got from corollary 2.1.1. Fortunately, the fake edge incident to a ∈ A can have any colour on its A side even when we fix the colour on its B side. This is because the edge distortion on a fake edge is not fixed. (since we artificially added that edge to the graph). So we shall change the edge distortion to force a colour on the A side of this fake edge to avoid a clash if needed.

Thus we have now freed up the colour σ_e^{-1}(x) for the A side of the edge e. Thus we can now colour the edge e with σ_e^{-1}(x) on its A side (and thus with colour x on its B side) without disrupting the proper distortion colouring of G' as required.

Now we need to find a way to keep adding the real edges in M indefinitely. The worst case scenario is clearly illustrated in this example graph below:



Suppose we managed to add edges e_1 and e_2 to G' without disrupting the edge distortion colouring property. Now suppose we also try to add the edge e_3 to G'. For the vertex a_3 ∈ A both real edges incident to a_3 are incident to 3 more real edges on their B side instead of 2. (including the blue edges e_1 and e_2 as real edges). Thus we will we need to adapt our argument accordingly for this more complicated situation.

Firstly observe that there are 3 black edges incident to e_3 on its B side. Hence there are 2 distinct colours α,β ∈ {0,1,2,3,4} available for the B side of e_3.

Now without loss of generality consider the 'upper' black edge incident to a_3 ∈ A. Observe that on its B side, i.e. at b_1 ∈ B, that edge is incident to 3 more real edges including e_1. Now here comes the trick. Consider a 2-factor of G'. Call it $\mathcal{M}$. This 2 factor $\mathcal{M}$ obviously covers each vertex in $G' \cup$ {e$_1$,e$_2$,e$_3$}and is a collection of vertex disjoint cycles. It is also a subgraph of G' and thus only consists of black and green edges. Since there are 2 black edges incident to b_1 ∈

B (not including the aforementioned 'upper' black edge) at least one of those 2 black edges is contained in the 2-factor $\mathcal{M}$. (Remember that none of the blue edges are contained in G' and thus are also not contained in $\mathcal{M}$). Without loss of generality let the 'bottom' black edge be contained in $\mathcal{M}$.

Now recall that in the proof of theorem 3.1 in [1] only 1 edge in each cycle cannot be pre-coloured arbitrarily. However when the number of colours allowed for the distortion colouring is 5 instead of 4, then there are 2 colours that can be assigned to the B side of that edge without disrupting the distortion colouring.(Refer to the proof in the appendix as to why is this is true). This means that there are 2 colours available for the B side of the 'bottom' black edge mentioned earlier without disrupting the distortion colouring. Thus there are now 3 distinct colours x,y,z $\in \{0,1,2,3,4\}$ available for the B side of the 'upper' black edge. Those colours correspond to σ^{-1}(x), σ^{-1}(y) and σ^{-1}(z) on the A side of that edge.

We need at least one of $\sigma_{e_3}^{-1}(\alpha)$, $\sigma_{e_3}^{-1}(\beta)$ to be available at $a_3 \in$ A so that we can use one of those colours to colour the A side of the edge e_3. (Recall that these are the only 2 colours for the A side of e_3 that will not disrupt the distortion colouring!)

So let us consider the edges incident to $a_3 \in$ A. The fake edge's permutation is not fixed and hence that edge does not fix a colour. The lower black edge a_3 $\in$ A does fix a colour. In the worst case scenario it 'occupies' either $\sigma_{e_3}^{-1}(\alpha)$ or $\sigma_{e_3}^{-1}(\beta)$. Without loss of generality assume that it occupies $\sigma_{e_3}^{-1}(\alpha)$. However the upper black edge can be coloured σ^{-1}(x), σ^{-1}(y) or σ^{-1}(z). One of which must be neither the colour used by the lower black edge nor $\sigma_{e_3}^{-1}(\beta)$. Hence $\sigma_{e_3}^{-1}(\beta)$ is available for the A side of the blue edge e_3. Thus we can colour e_3 without disrupting the distortion colouring!

Hence we can add all of the 3m real edges in M to G' without disrupting the distortion colouring. Then we can simply remove the vertices in P (dummy vertices) and the fake edges. (removing edges and vertices will preserve the distortion colouring). What remains is then the original graph G with a proper distortion colouring using 5 colours as required. $\square$

References

[1] Dr Agelos Georgakopoulos. Delay colourings of cubic graphs
[2] R. Diestel. Graph Theory (3rd edition). Springer-Verlag, 2005. Electronic edition available at: http://diestel-graph-theory.com/index.html

Appendix

Corollary 2.1.2. *For every 3 regular bipartite multigraph G, and any edge distortions, there is a proper distortion-colouring of E(G) with 5 colours 0,1,2,3,4. (Please note that the proof is almost identical to the proof in [1, Theorem 3.1] with a few minor adjustments for the fact that 5 colours are used instead of 4. Moreover the notation used here is slightly different than in the rest of the paper as the proof follows the notation from [1]).*

Proof: It is well known that the edges of a regular bipartite multigraph can be decomposed into disjoint perfect matchings. So let M, M', and M'' be perfect matchings of G with M $\cup$ M' $\cup$ M'' = E(G). Then M' $\cup$ M'' is a 2 factor, and it can be decomposed into a collection $\mathcal{C}$ of vertex disjoint cycles.

Let {A,B} be the bipartition of V(G). We are going to let each element of $\mathcal{C}$ choose the colours of edges of M incident with its A side. More precisely, given a C $\in \mathcal{C}$, let $M_{C\cap A}$denote the set of edges M incident with $C \cap A$. We are going to prove that:

For every C $\in$ $\mathcal{C}$ there is a 5 colouring f_A of $M_{C\cap A}$ such that for every 5 colouring f_B of $M_{C\cap B}$, there is a 5-colouring f_C of E(C) such that $f_A \cup f_B \cup f_C$ is a proper distortion-5-colouring. (1)

Note that (1) easily implies a proper distortion-colouring of E(G) with 5 colours: the sets $M_{C\cap A}$ | C $\in \mathcal{C}$ are pairwise edge disjoint, and their union is M. Thus we can begin by colouring each of them by a colouring f_A as in (1) and then we can extend the colouring to each C $\in \mathcal{C}$ keeping it proper.

So let use prove (1). Given C $\in \mathcal{C}$, pick a 2-edge subarc uvy of C with u,y $\in$ A. Distinguish two cases:

If the permutations of the edges uv, vy are identical, then give the edges m_u, m_y of M incident with u,y colours that are different (when seen from A). If C happens to be a 2-cycle, in which case u = y give $m_u = m_y$ any colour.

If those permutations are not identical, then colour (the A side) of both m_u,m_u with a colour α such that vu(α) $\neq$ vy(α) (i.e the colours on their B sides are not the same)

In both case, colour the rest of $M_{C\cap A}$ arbitrarily; those colours will not matter.

We claim that this colouring f_A has the desired property. To prove this, let f_B be any colouring of $M_{C\cap B}$, and note that for every edge e $\in$ E(C) the set L_e of still available colours for e, that is, the colours that would not conflict with f_A $\cup$ f_B if given to e on its B side, say, has at least 3 elements;indeed, only 2 edges adjacent with e have been coloured so far and we had 5 colours to begin with.

Let us first deal with the case where C is not a 2-cycle, and consider again the two edges vu, vy, and so $L_{vu} \neq L_{vy}$ are neither equal nor disjoint, and each contains at least three colours. We can find 2 common colours $\zeta, \eta \in L_{vu} \cap L_{vy}$ and another 2 colours γ in L_{vu} and δ in L_{vy}. So that they are all distinct. Now colour vu with uv(γ) (so that the colour seems to be γ on its B side), and note that our colouring is still proper, since this colour came from the allowed list. Consider the next edge ux of C incident with u. This edge still has at least 2 allowed colours, after we colour uv (recall that $|L_e| \geqslant 3$). so give it one of those colours. Continue like this along C, to properly colour all its edges except the last edge vy. Now there are 2 options available for the last edge. Assigning one of those colours to vy Completes the proper distortion colouring of C.

if C is a 2- cycle then the situation is much simpler, and it is straightforward to check that (1) holds by distinguishing two cases according to whether its 2 edges bear the same permutations or not.

This completes the proof. Note that we have proved that the 'last' edge in a cycle can have 2 colours assigned to it for any proper edge distortion colouring.

□