Γενικεύοντας από πεπερασμένα γραφήματα σε άπειρα· και ακόμα παραπέρα

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Universität Hamburg

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Things that go wrong in infinite graphs

Many finite theorems fail for infinite graphs:

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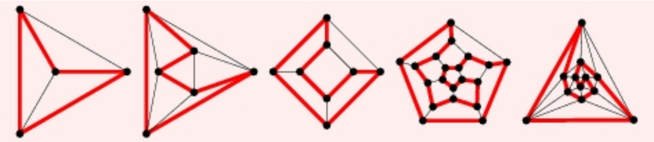
Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems
- Extremal graph theory
- Cycle space theorems
- many others ...

Hamilton cycle: A cycle containing all vertices.

Some examples:



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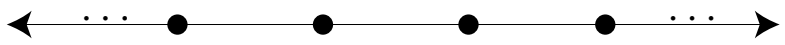
$\Rightarrow$ need more general notions

Spanning Double-Rays

Classical approach to ‘save’ Hamilton cycle theorems: accept double-rays (διπλές ακτίνες) as infinite cycles



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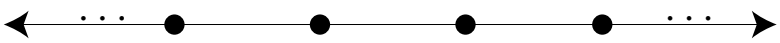

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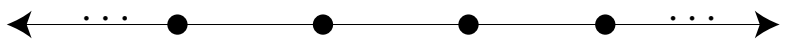

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Theorem (Tutte '56)

Every finite 4-connected planar graph has a Hamilton cycle

4-connected := you can remove any 3 vertices and the graph remains connected

Spanning Double-Rays

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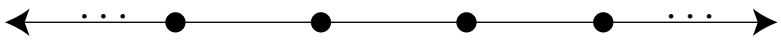

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Theorem (Yu '05)

Every locally finite 4-connected planar graph has a spanning double ray ...

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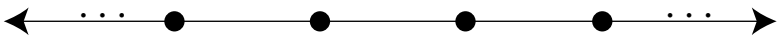

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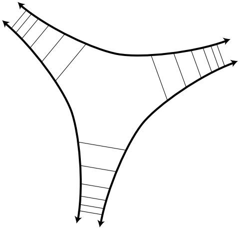
Theorem (Yu '05)

Every locally finite 4-connected planar graph has a spanning double ray ... unless *it is* 3-divisible (τριχοτομίσιμο).

Compactifying by Points at Infinity

A 3-divisible graph



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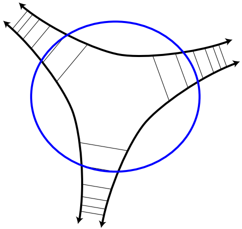

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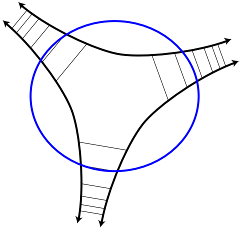
A 3-divisible graph
can have no spanning double ray



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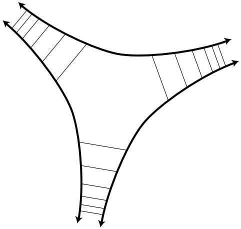

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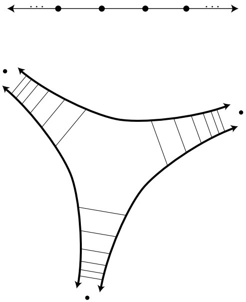

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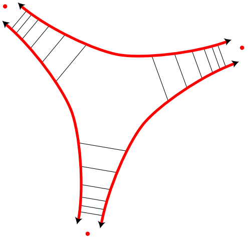

... but a Hamilton cycle?

Ends

πέρας (end): equivalence class of rays

two rays are equivalent if no finite vertex set separates them

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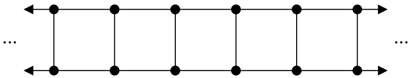

two ends

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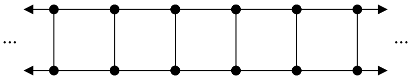

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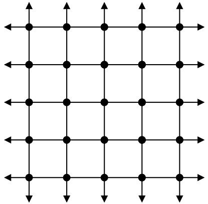

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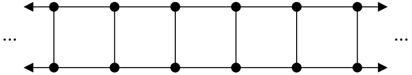

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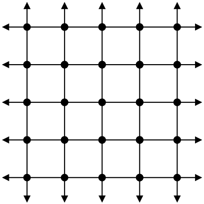

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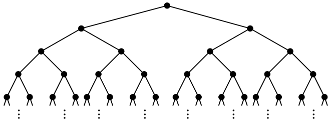
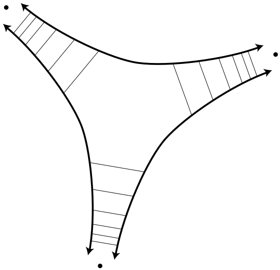


uncountably many ends

The End Compactification



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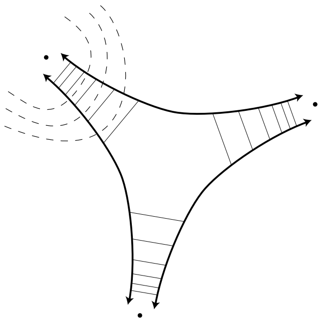

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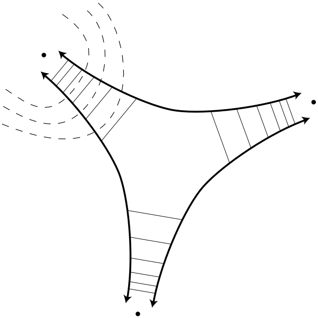
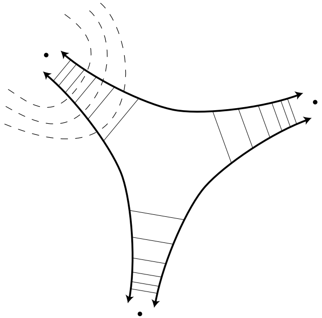


Every ray converges to its end

The End Compactification

$$|G|$$

= end compactification = Freudenthal compactification



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Denote by $|G|_\ell$ the completion of (G, d_ℓ)

Theorem (G '06)

If $\sum_{e \in E(G)} \ell(e) < \infty$ then $|G|_\ell$ is homeomorphic to $|G|$.

Infinite Cycles

Circle:
A homeomorphic image of S^1 in $|G|$.

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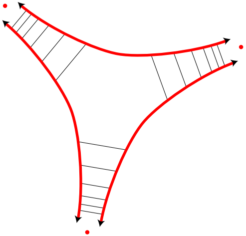

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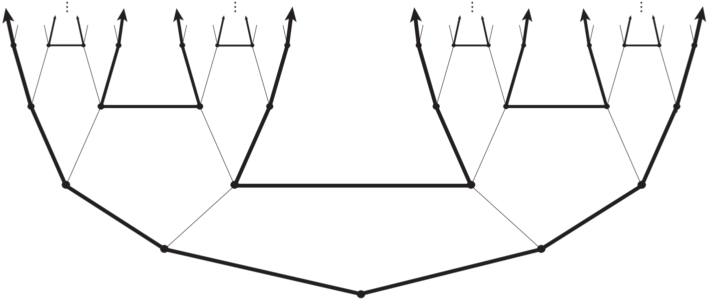

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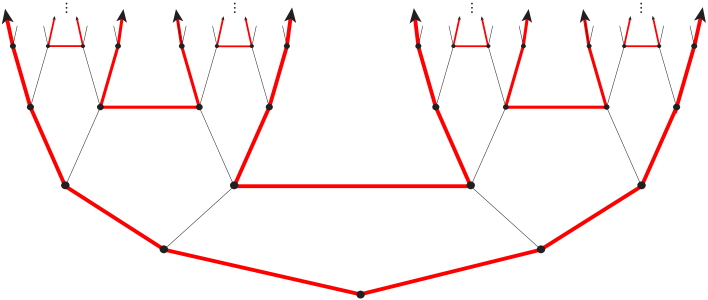

the wild circle of Diestel & Kühn

Fleischner's Theorem

Theorem (Fleischner '74)

The square of a finite 2-connected graph has a Hamilton cycle

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The square of a finite 2-connected graph has a Hamilton cycle

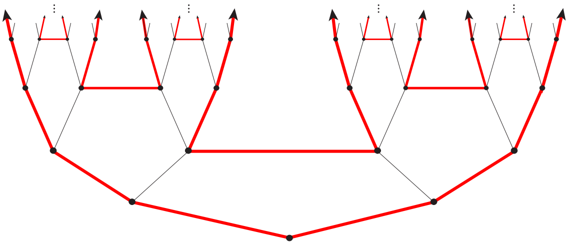
Theorem (Thomassen '78)

The square of a locally finite 2-connected 1-ended graph has a Hamilton circle (i.e a spanning double-ray).

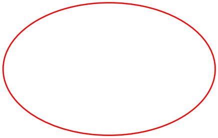
The Theorem

Theorem (G '06, *Adv. Math. '09*)

The square of any locally finite 2-connected graph has a Hamilton circle



Proof?



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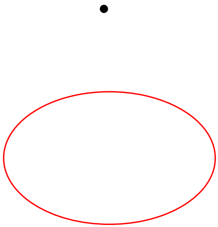

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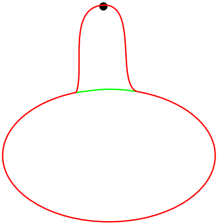

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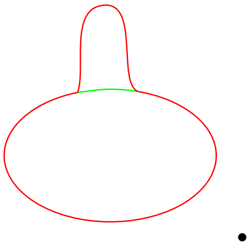

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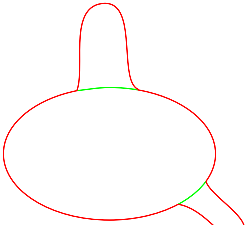

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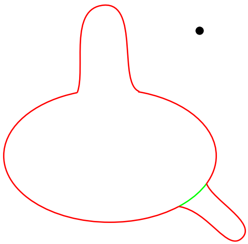

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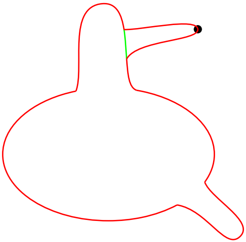

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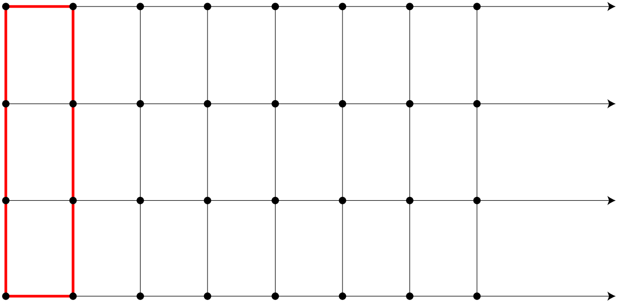

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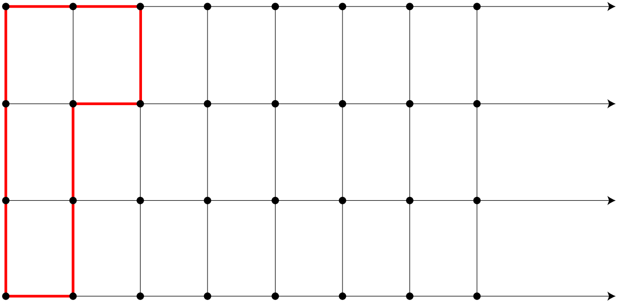

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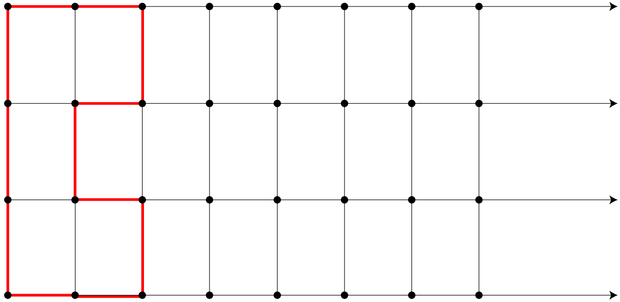

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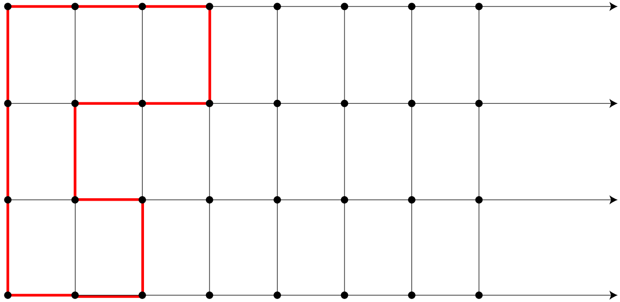

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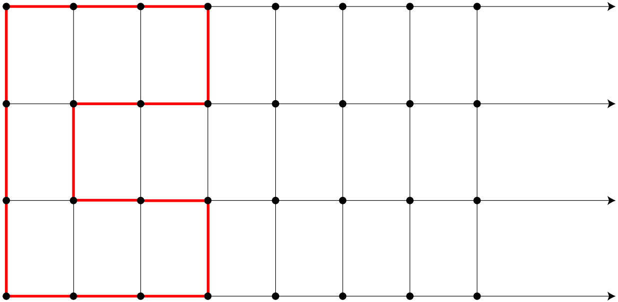

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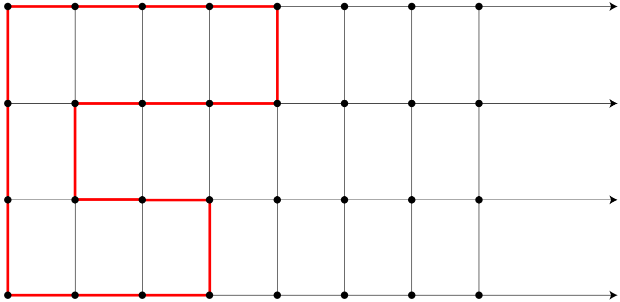

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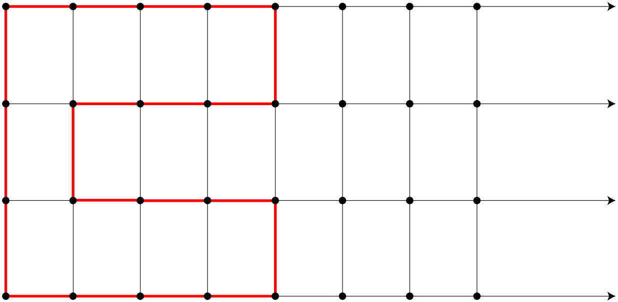

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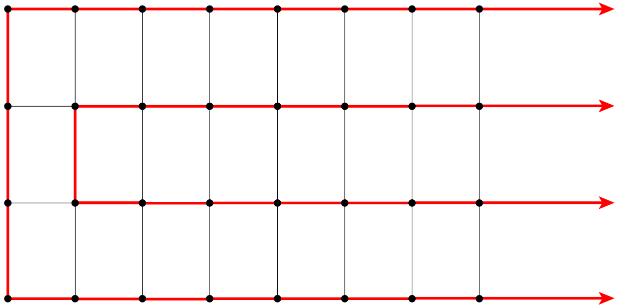

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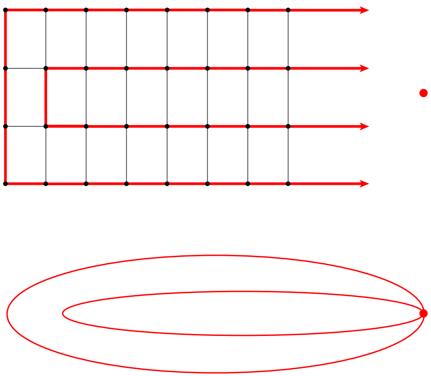
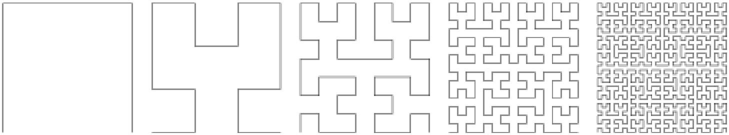


Proof?

Hilbert's space filling curve:

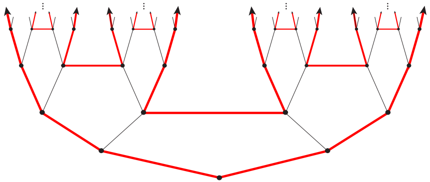


a sequence of injective curves with a non-injective limit

The Theorem

Theorem (G '06)

The square of any locally finite 2-connected graph has a Hamilton circle.



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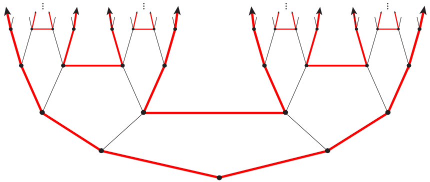

Corollary (informal)

Most Cayley graphs are hamiltonian.

Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser '59)

Does every finite connected Cayley graph have a Hamilton cycle?

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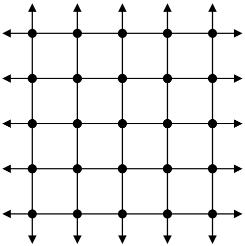

Hamiltonicity in Cayley graphs

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Does every connected 1-ended Cayley graph have a Hamilton circle?

Problem

Characterise the locally finite Cayley graphs that admit Hamilton circles.

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- many others ...

Cycle Space

The cycle space (κυκλόχωρος) $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$ (one coordinate per edge of G);
- Consists of all sums of edge-sets of cycles of G.

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- Allows edge sets of infinite circles;
- Allows infinite sums (whenever well-defined).

The topological Cycle Space

Known facts:

- A connected graph has an Euler tour iff every edge-cut is even (Euler)
- G is planar iff $\mathcal{C}(G)$ has a simple generating set (MacLane)
- The geodetic cycles of G generate $\mathcal{C}(G)$.

Generalisations:

Bruhn & Stein

Bruhn & Stein

G & Sprüssel

MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

simple: no edge appears in more than two generators.

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Theorem (Bruhn & Stein'05)

... verbatim generalisation for locally finite G

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Theorem (Diestel & Sprüssel' 09)

$\mathcal{C}(G)$ coincides with the first Čech homology group of $|G|$ but not with its first singular homology group.

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Problem

Can we use concepts from homology to generalise theorems from graphs to other topological spaces?

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Theorem (G '09)

...the cycle decomposition theorem for graphs generalises to arbitrary continua if one considers the 'right' homology...

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Let R be a ring and E any set
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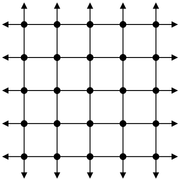

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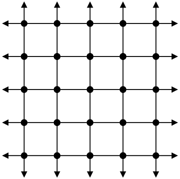

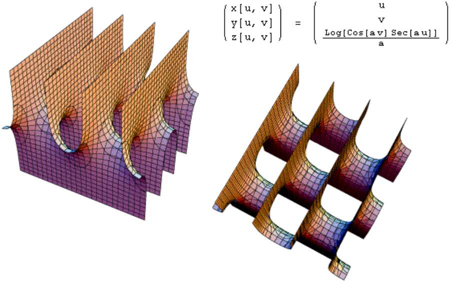

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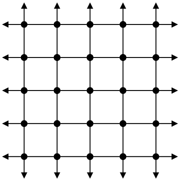

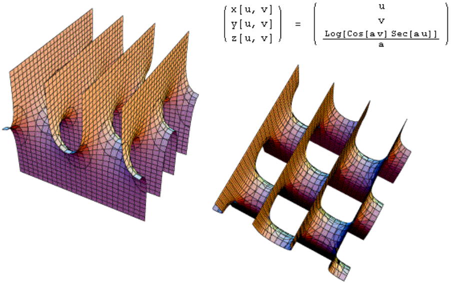

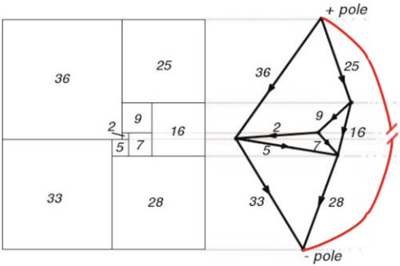

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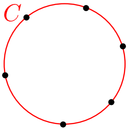

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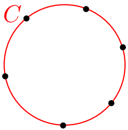

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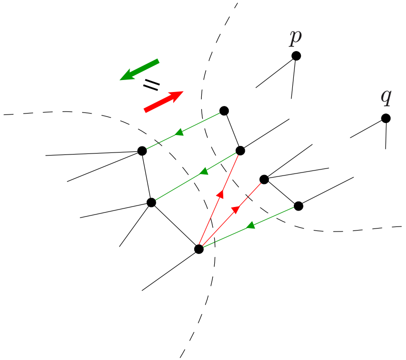
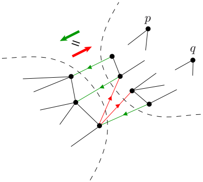
Finite Networks	Networks of finite total resistance	Infinite Networks
Unique solution	?	Not necessarily unique solution

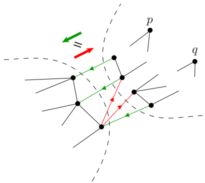
Good flows

Good flow:
The net flow along any such cut must be zero:



The Theorem





Finite Networks	Networks of finite total resistance	Infinite Networks
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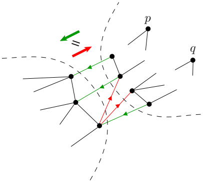
Theorem (G '08)

In a network with $\sum_{e \in E} r(e) < \infty$ there is a unique good flow with finite energy that satisfies Kirchhoff's second law.



Energy of f: $\frac{1}{2} \sum_{e \in E} f^2(e) r(e)$

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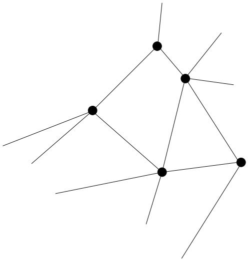

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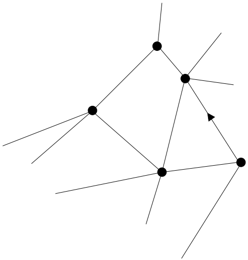

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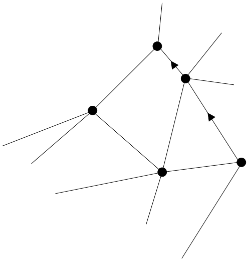

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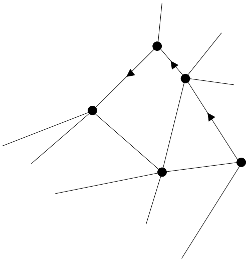

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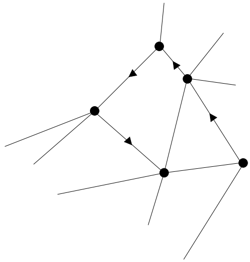

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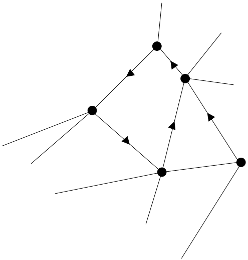

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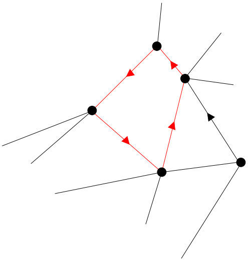

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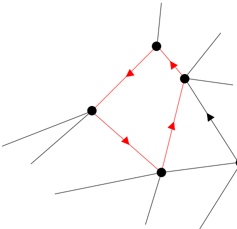

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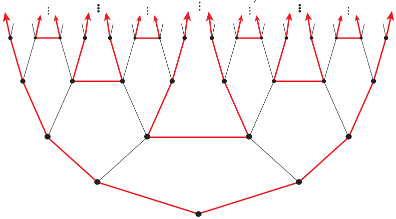

Finding wild circles by a limit construction

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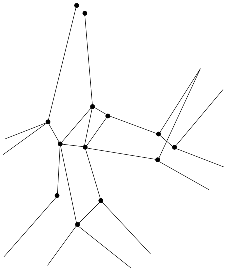

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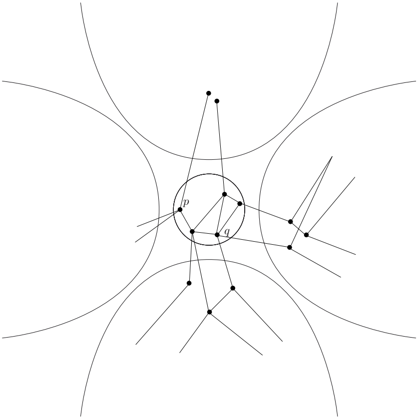

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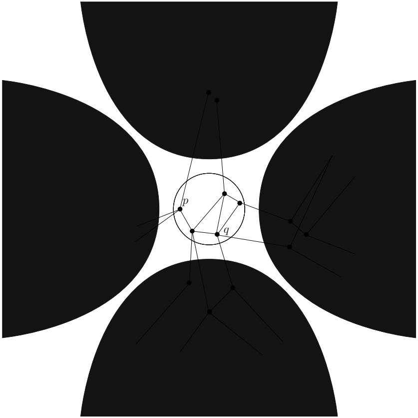

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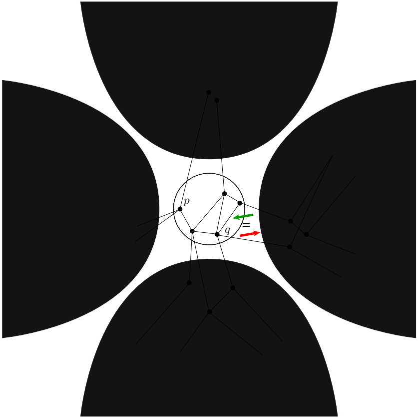

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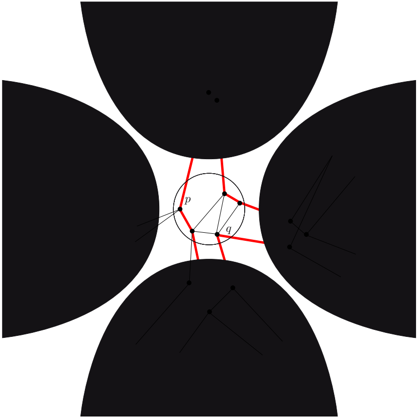

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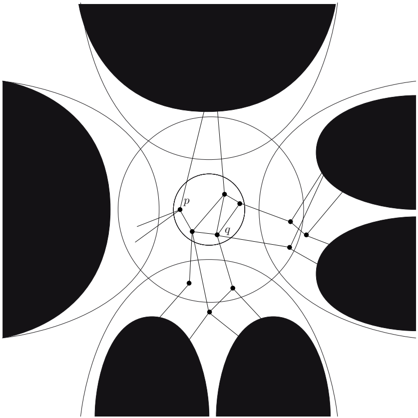

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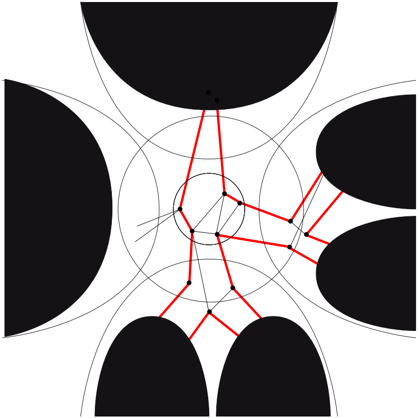

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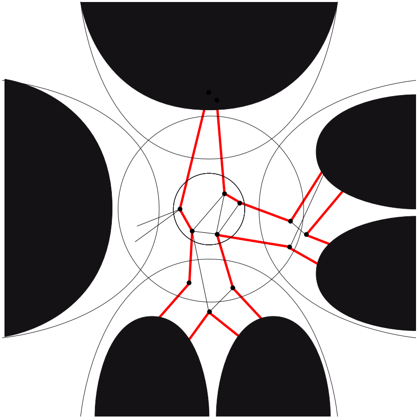

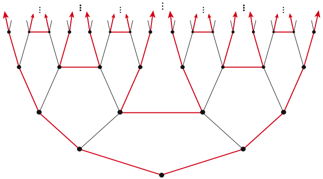

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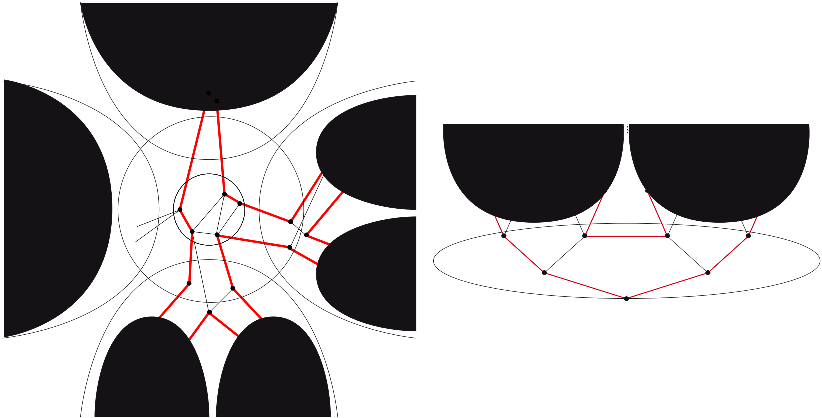

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