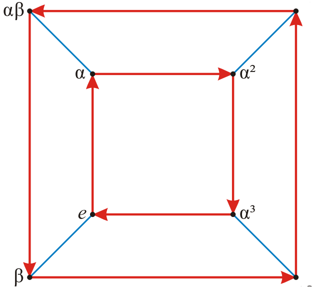
The planar cubic Cayley graphs

Agelos Georgakopoulos

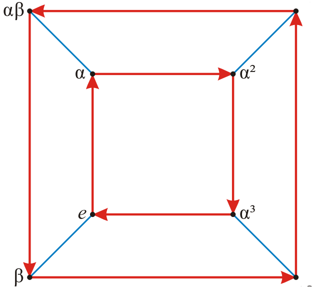
Technische Universität Graz

Berlin, 1.11.10

Cayley graphs



$$\langle \alpha, \beta \mid, \beta^2, \alpha^4, (\alpha\beta)^2 \rangle$$



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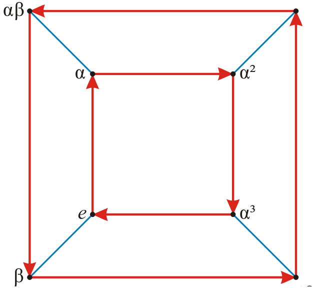

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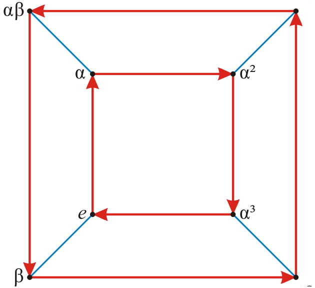

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$$g \xrightarrow{s} gs$$

Sabidussi's Theorem

Theorem (Sabidussi's Theorem)

A properly *edge-coloured digraph is a Cayley graph iff for every $x, y \in V(G)$ there is a colour-preserving automorphism mapping x to y.*

properly edge-coloured := no vertex has two incoming or two outgoing edges with the same colour

---

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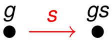
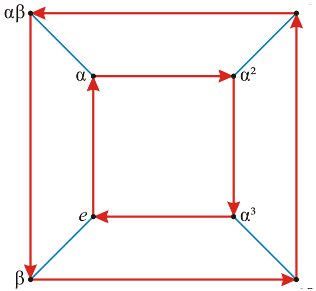


Charactisation of the finite planar groups

Theorem (Maschke 1886)

Every finite planar group is a group of isometries of S^2.



The Cayley complex

Let $\Gamma = \langle a, b, c, \ldots \mid R_1, R_2 \ldots \rangle$ be a group presentation. Define the corresponding Cayley complex $CC\langle a, b, c, \ldots \mid R_1, R_2 \ldots \rangle$ by:

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Given a planar Cayley graph, can you find a presentation in which the relators induce precisely the face boundaries?

Yes! :

Theorem (Whitney '32)

Let G be a 3-connected plane graph. Then every automorphism of G extends to a homeomorphism of the sphere.

Proving Maschke's Theorem

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Let X be the corresponding Cayley complex.

Then X is homeomorphic to S^2. Thus:

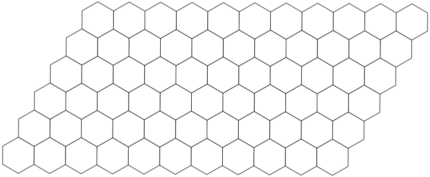
Theorem (Maschke 1886)

Every finite planar group is a group of isometries of S^2.

The 1-ended planar groups

Theorem ((classic) Macbeath, Wilkie, ...)

Every 1-ended planar Cayley graph corresponds to a group of isometries of $\mathbb{R}^2$ or $\mathbb{H}^2$.



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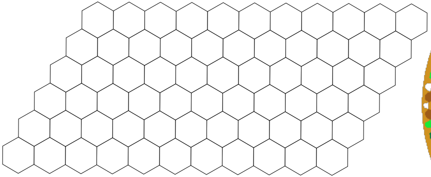

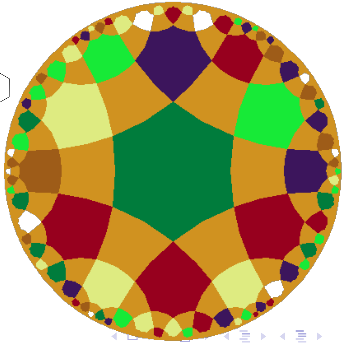

Planar groups and fundamental groups of surfaces

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Theorem (G '10)

A group has a planar Cayley complex if and only if it has a VAP-free Cayley graph.

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Classification of the cubic planar Cayley graphs

Theorem (G '10)

Let G be a planar cubic Cayley graph. Then G is colour-isomorphic to precisely one element of **the list***.*

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Let G be a planar cubic Cayley graph. Then G is colour-isomorphic to precisely one element of **the list**.
Conversely, for every element of the list and any choice of parameters, the corresponding Cayley graph is planar.

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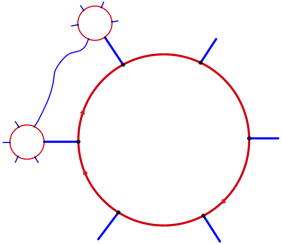
Examples



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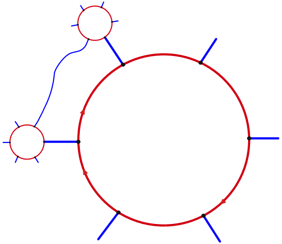

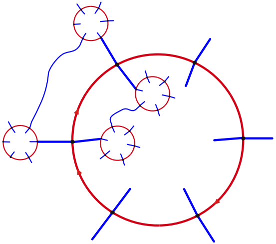

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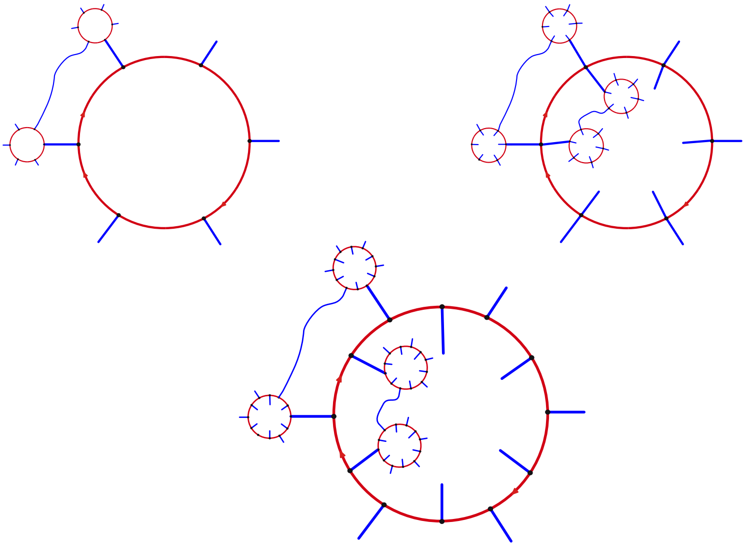

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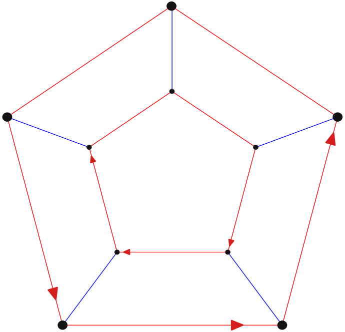

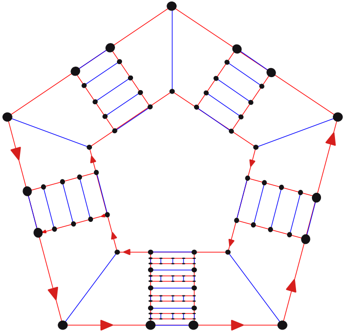

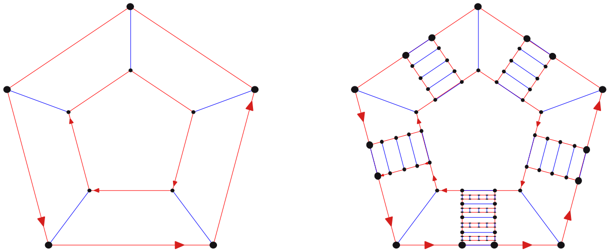

Corollary (G '10)

Every planar cubic Cayley graph has an almost planar Cayley complex.

Cayley graphs without finite face boundaries

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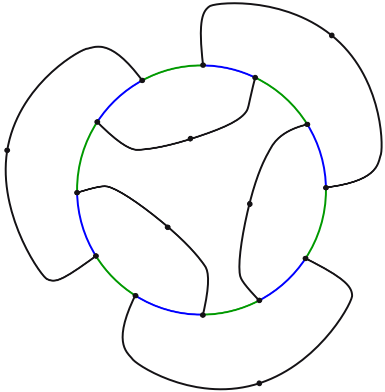

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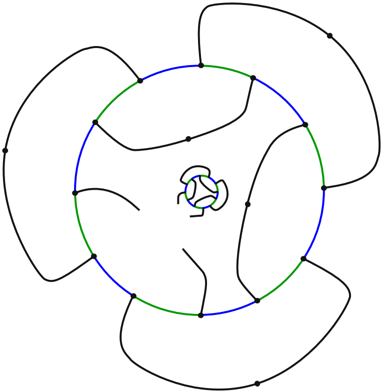

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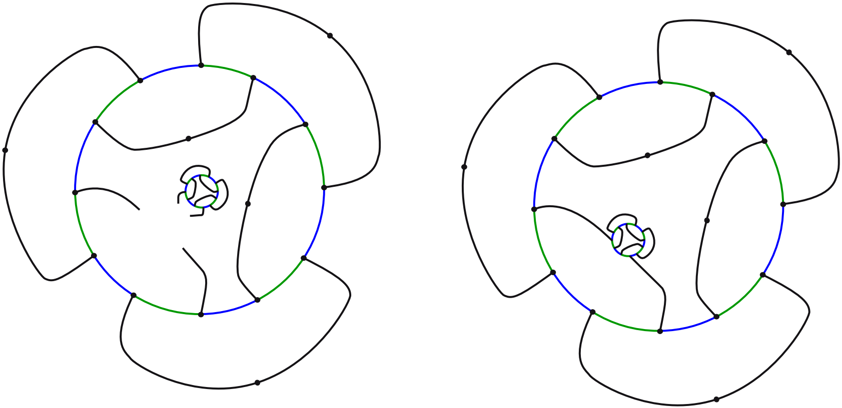

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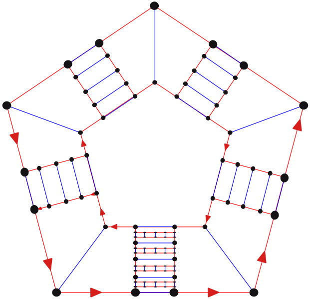

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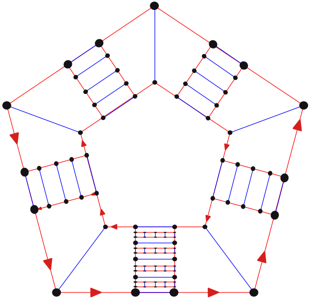
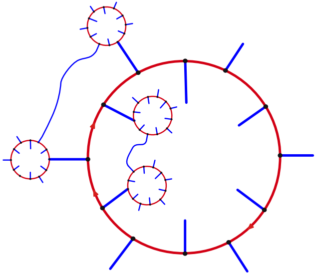
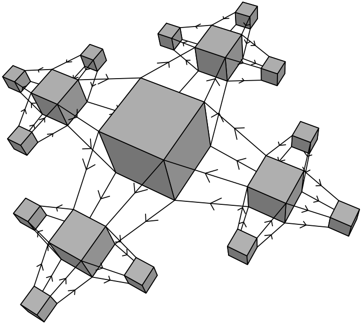




Stallings' Theorem

Theorem (Stallings '71)

Every group with >1 ends can be written as an HNN-extension or an amalgamation product over a finite subgroup.



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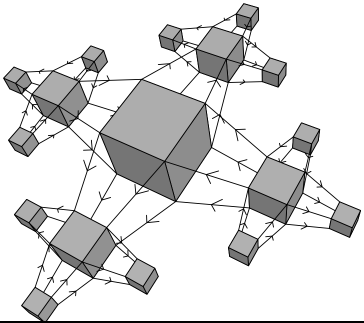

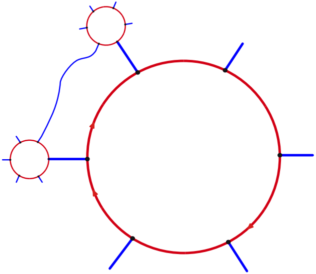

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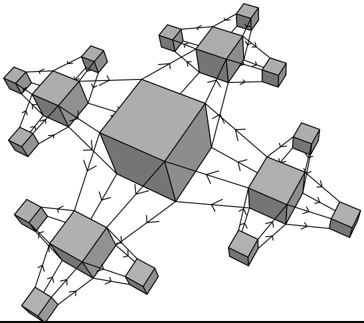
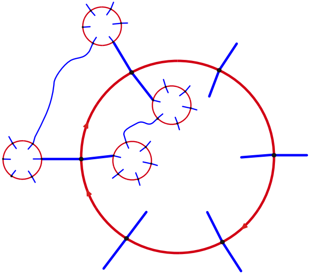
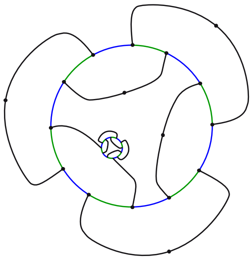




Group splittings by topological minors



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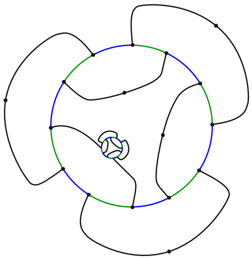

Conjecture

Let $G = Cay(\Gamma, S)$ be a Cayley graph with > 1 ends. Then there is a non-trivial splitting of G as a union of subdivisions of Cayley graphs.

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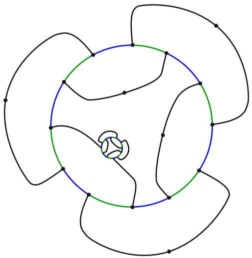

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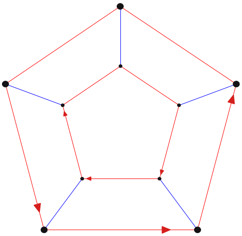

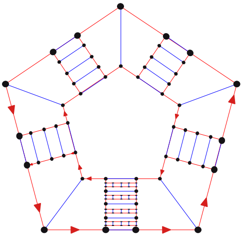

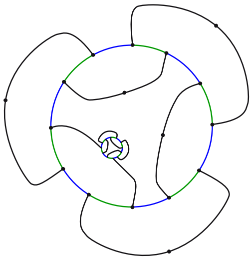

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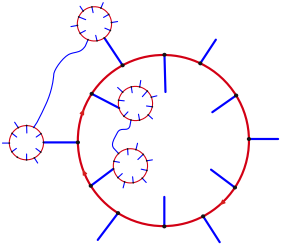

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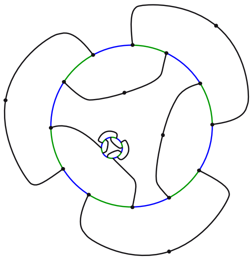

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Corollary (G '10)

True for planar cubic Cayley graphs.