Every rayless graph has an unfriendly partition

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Technische Universität Graz
and
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Universität Hamburg

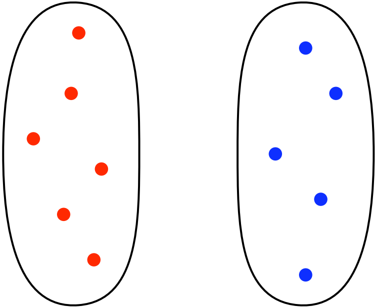
EuroComb '09, 9.9.9

Joint with Henning Bruhn, Reinhard Diestel and Philipp Sprüssel

The Unfriendly Partition Conjecture

Conjecture

Every countable graph admits a bipartition of its vertices such that every vertex has at least as many opponents as it has friends.



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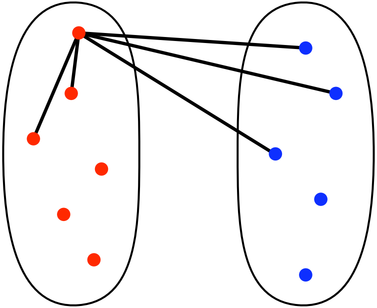

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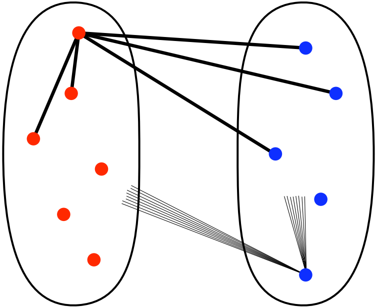

Finite graphs have UFP's

Theorem

Every finite graph has an unfriendly partition

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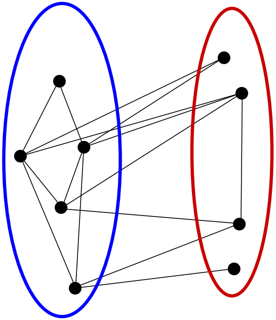
proof: consider a cut maximising the number of cross-edges



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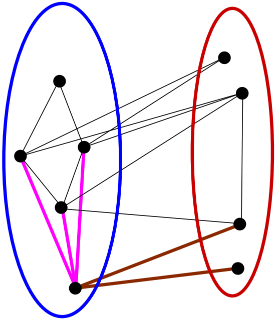

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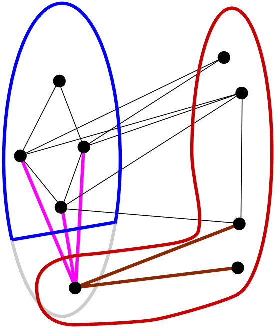

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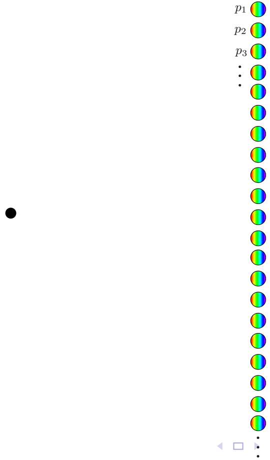

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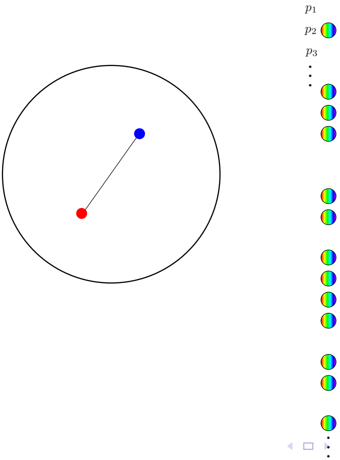

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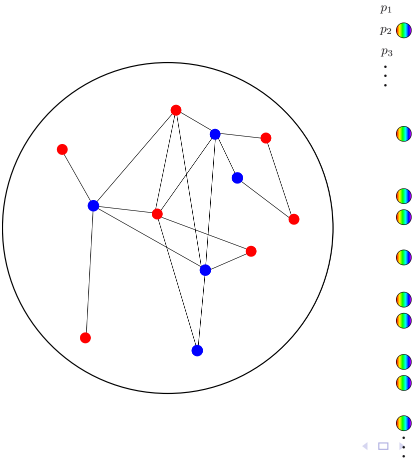

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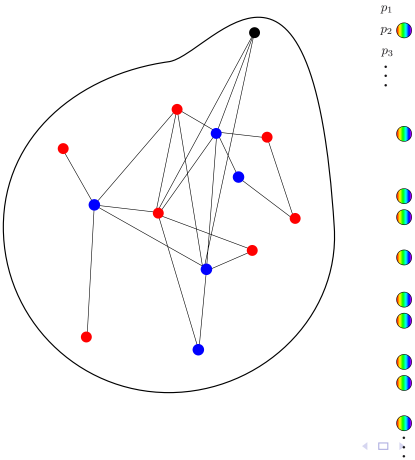

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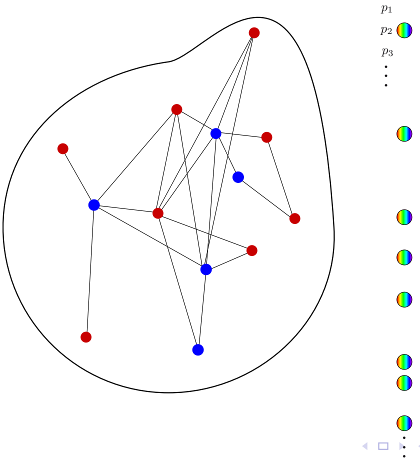
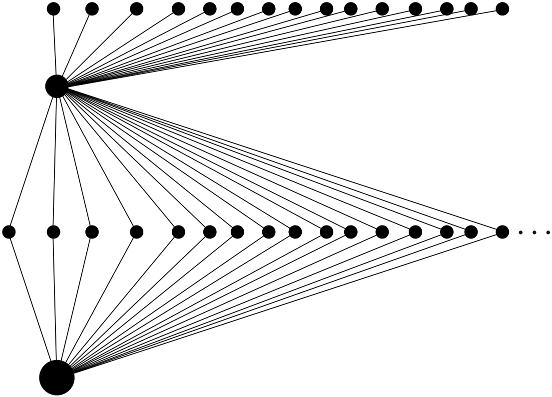


Non-locally-finite graphs

How does the argument fail if G has vertices of infinite degree?



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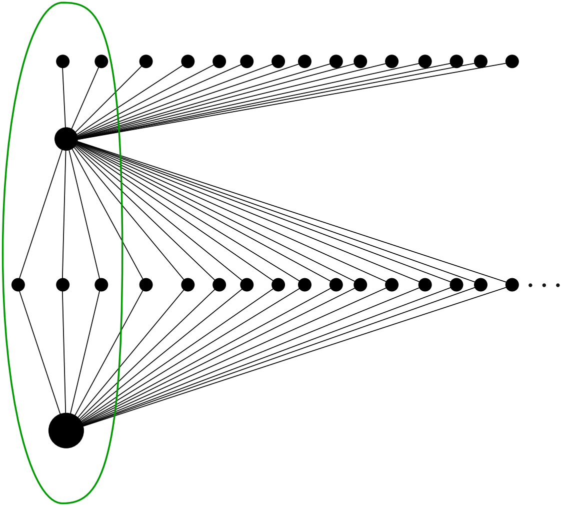

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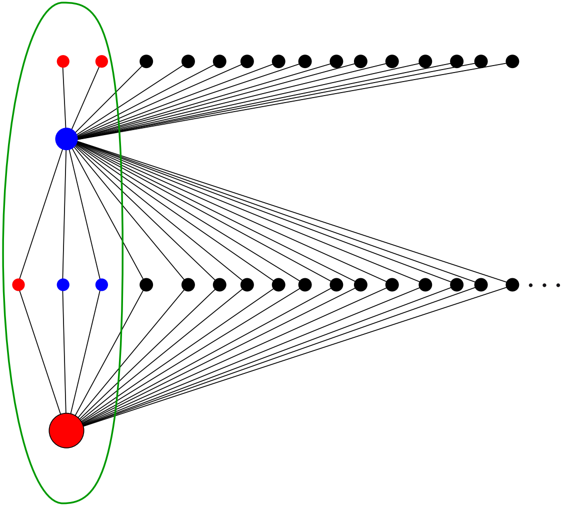

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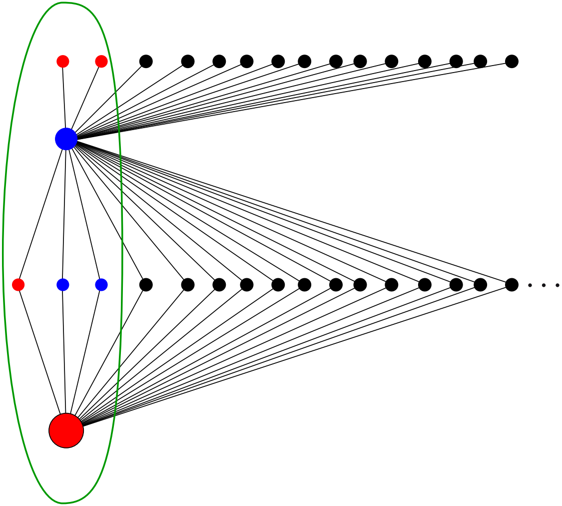

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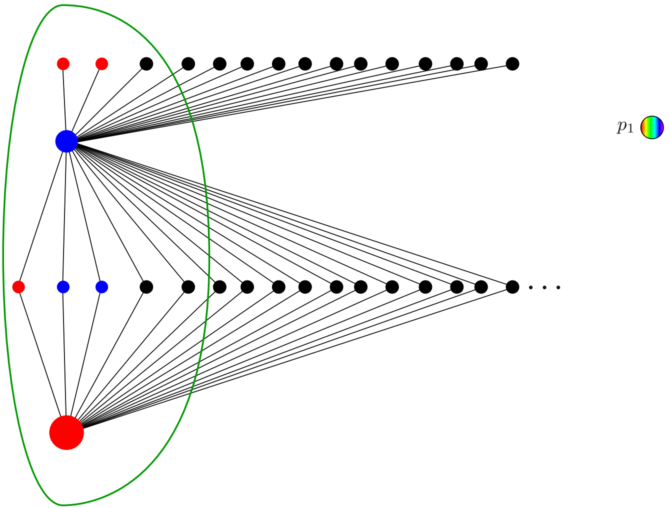

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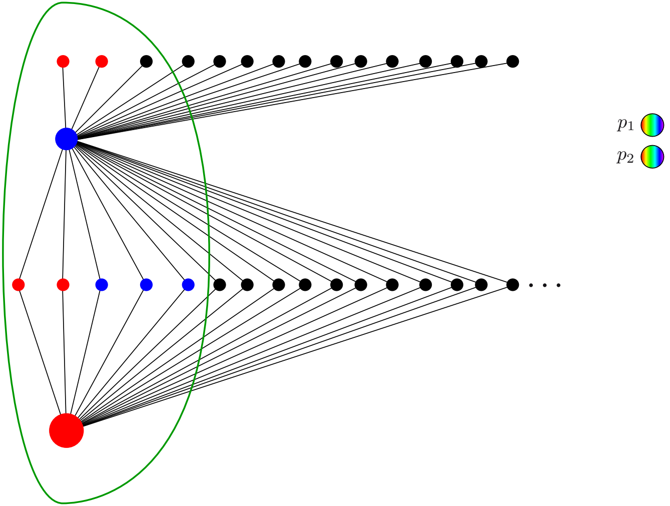

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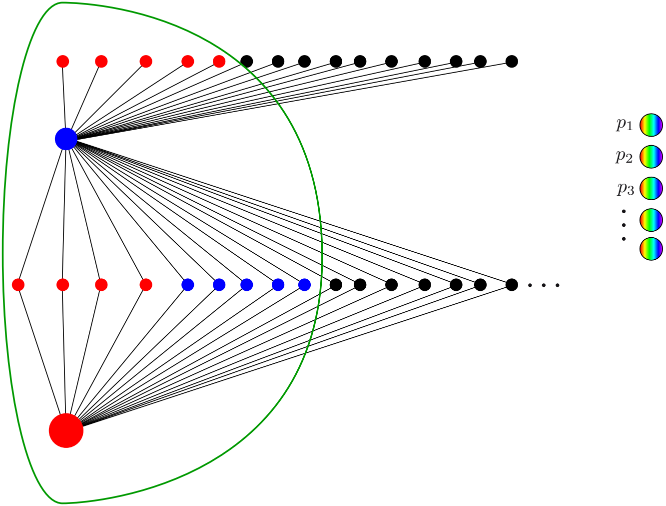

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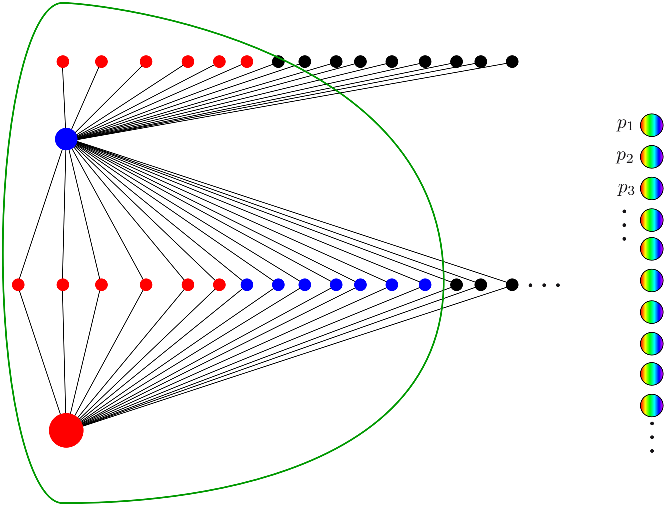

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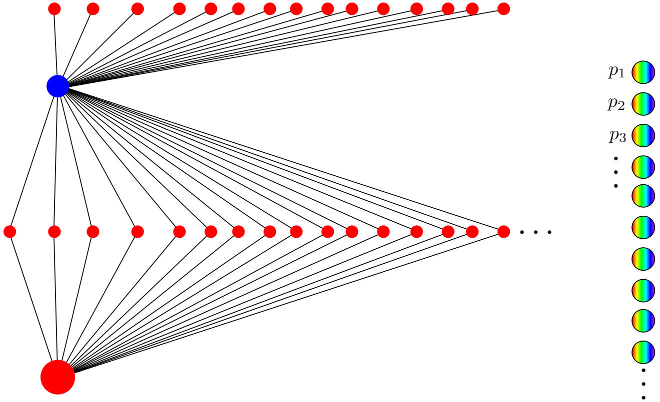

Known facts

- If G has only finitely many vertices of infinite degree then it admits an unfriendly partition (Aharoni, Milner, and Prikry '90)

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i.e. contains no infinite paths

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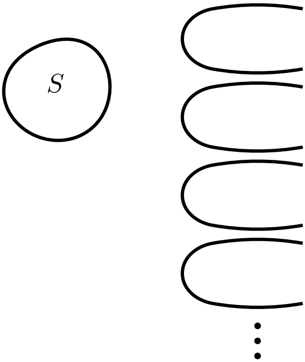

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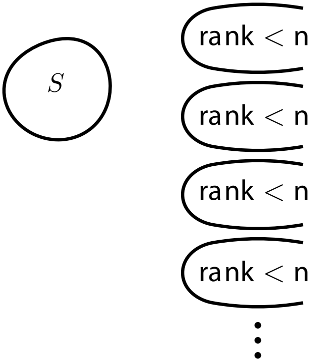
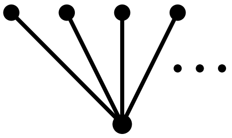


Examples



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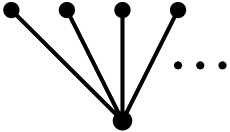

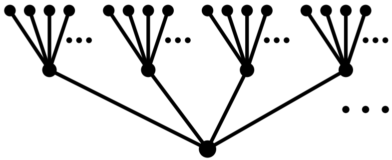

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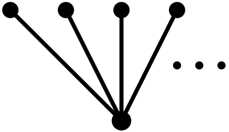

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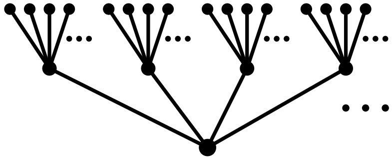

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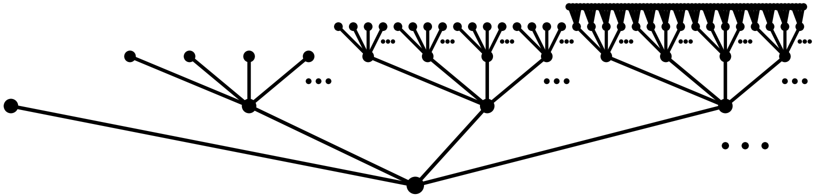

rank ω

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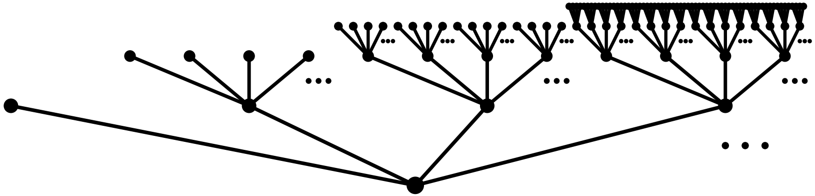

rank ω

Theorem (Schmidt '83)

A graph has a rank iff it is rayless.

Proof

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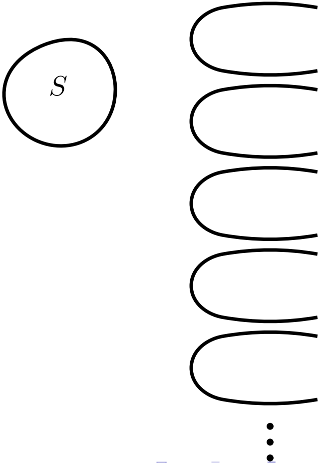

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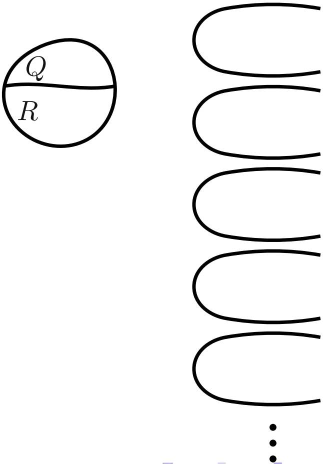

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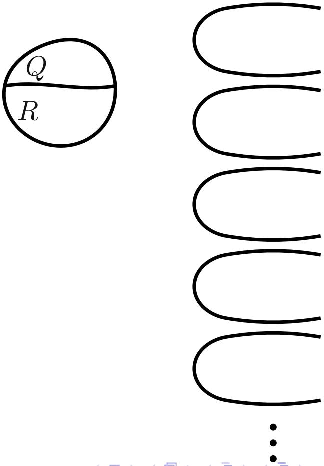

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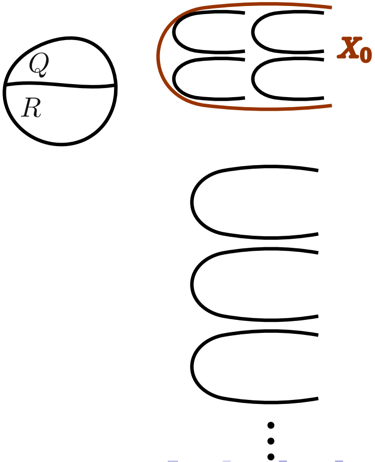

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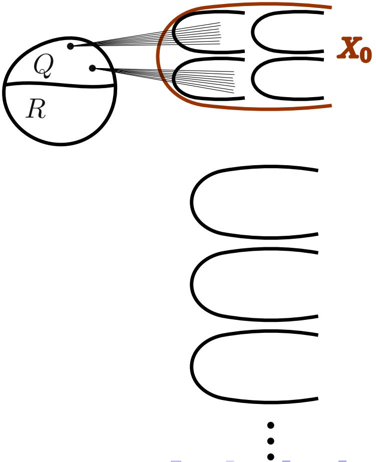

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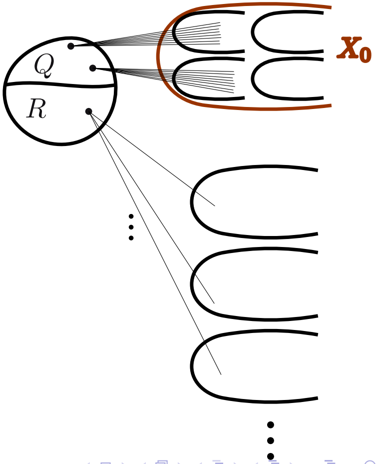

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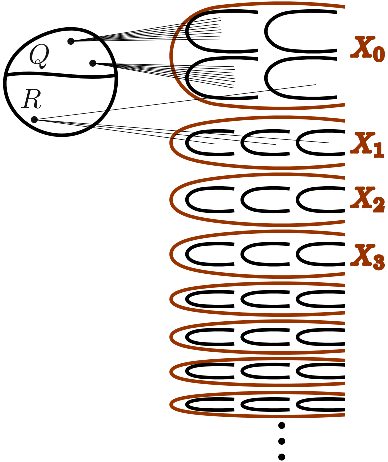

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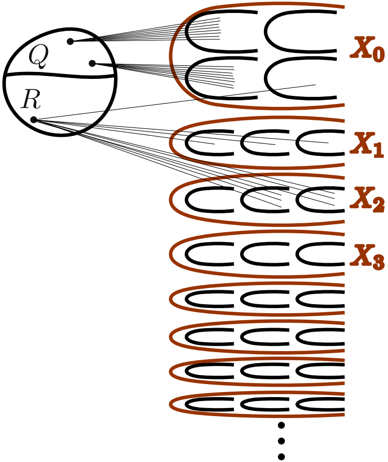

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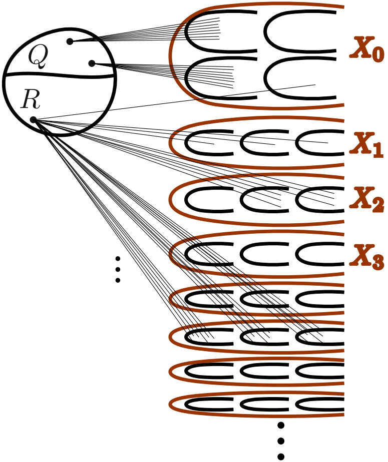

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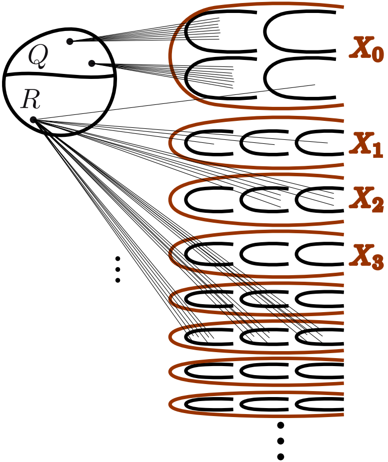

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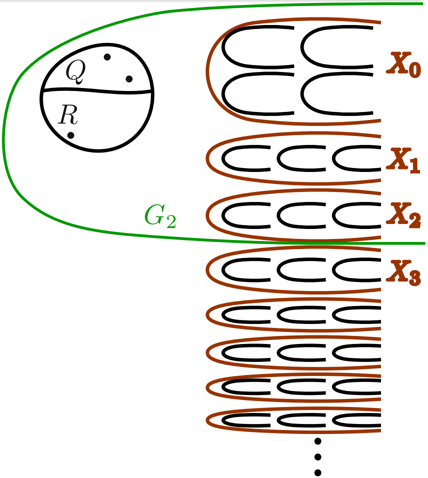

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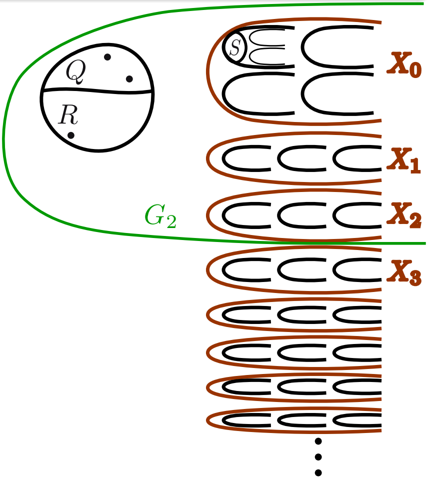

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p_2

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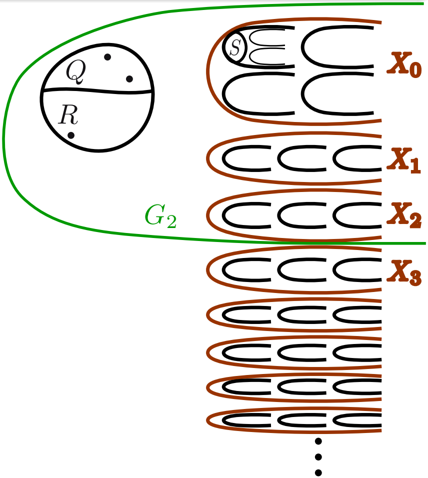

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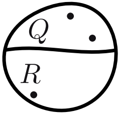

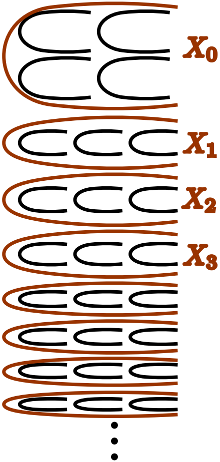

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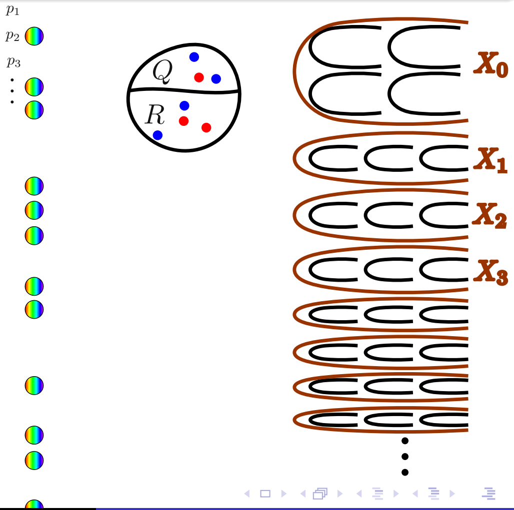

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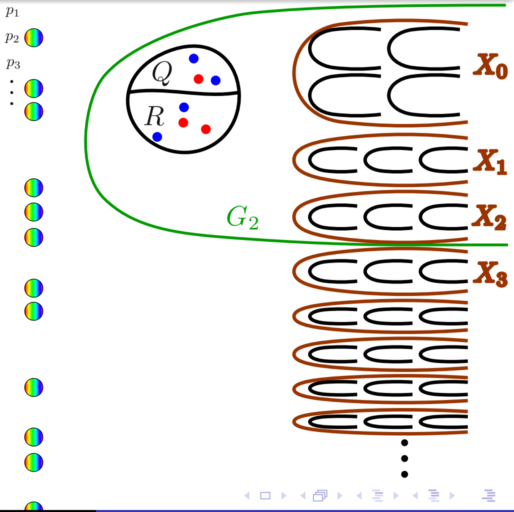

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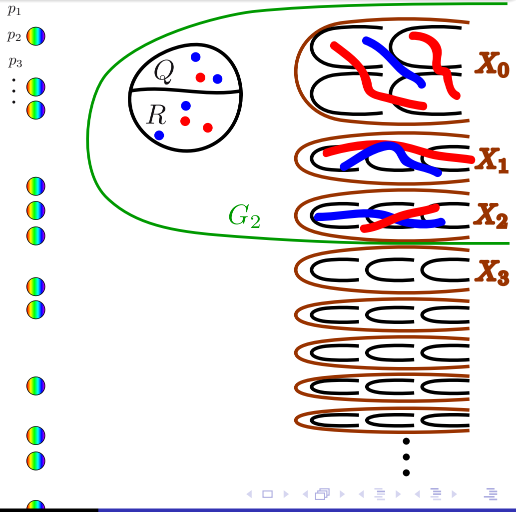

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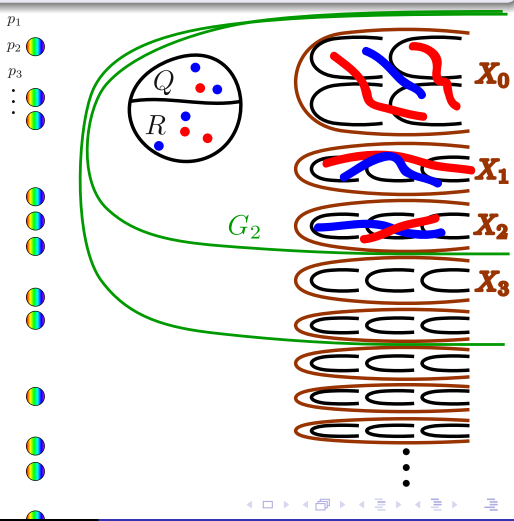
Q: vertices S that have their full degree in finitely many components

R: Other vertices of S

Vertices in R have many more neighbours in X_i than in $S \cup X_0 \cup \ldots \cup X_{i-1}$



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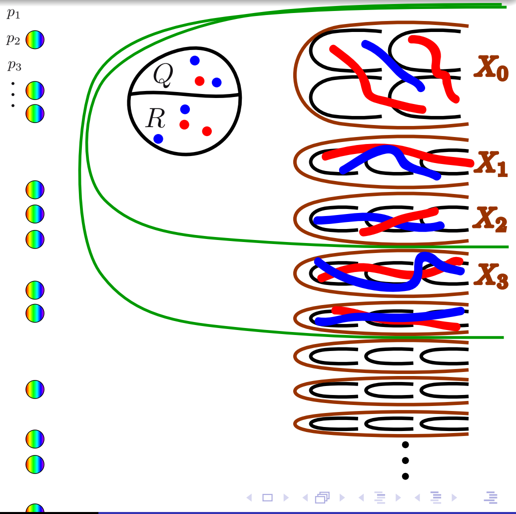

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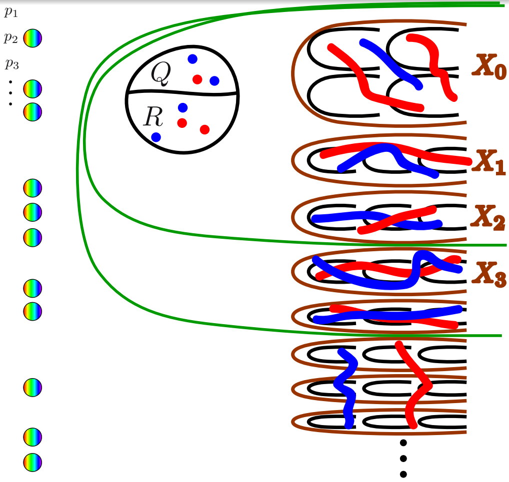

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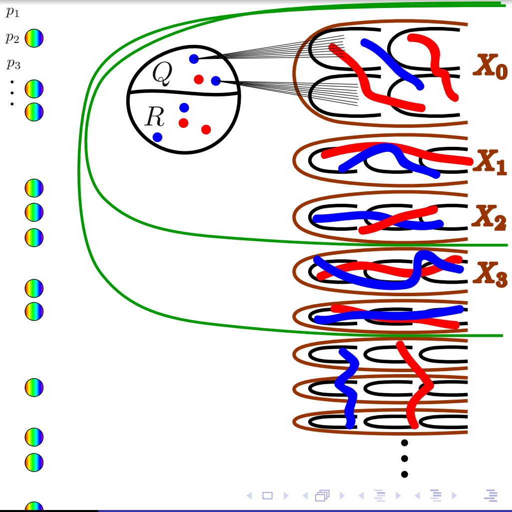

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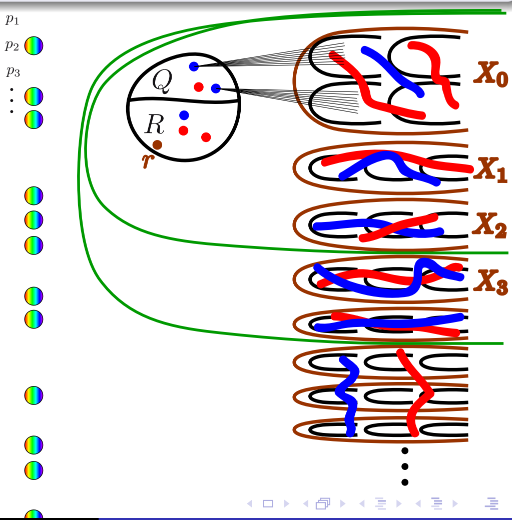

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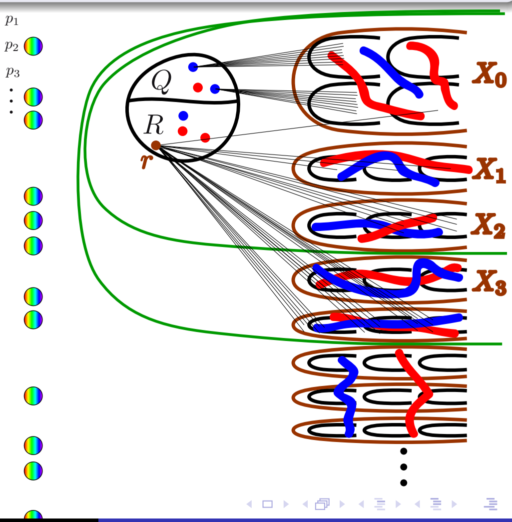

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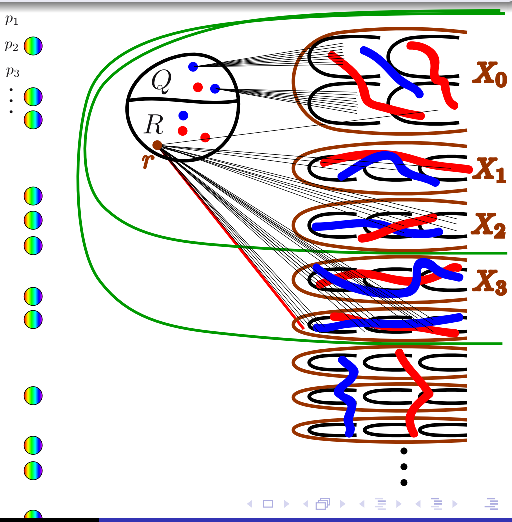

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