A new homology for infinite graphs and metric continua

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Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

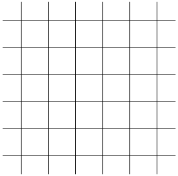
- A vector space over $\mathbb{Z}_2$
- Consists of all sums of cycles

i.e., the first simplicial homology group of G.

The topological cycle space $\mathcal{C}(G)$ of a locally finite graph G is defined similarly but:

- Allows edge sets of infinite circles;
- Allows infinite sums (whenever well-defined).

Let’s play



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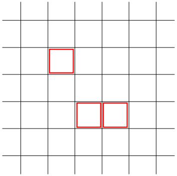

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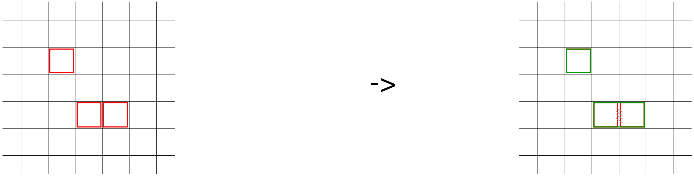

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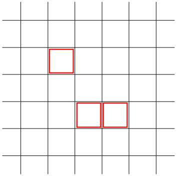

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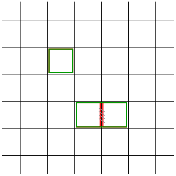

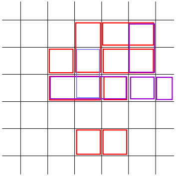

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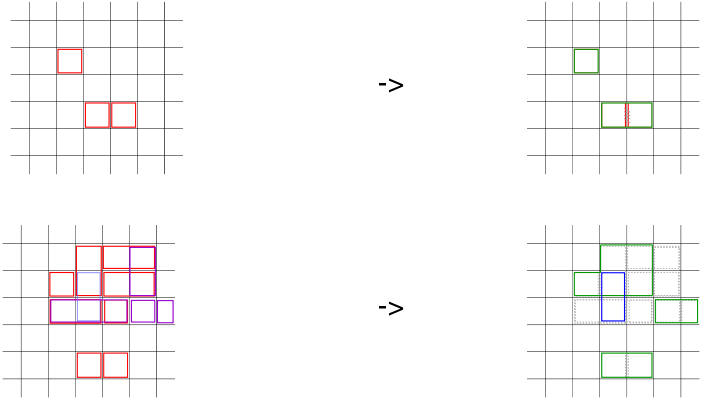

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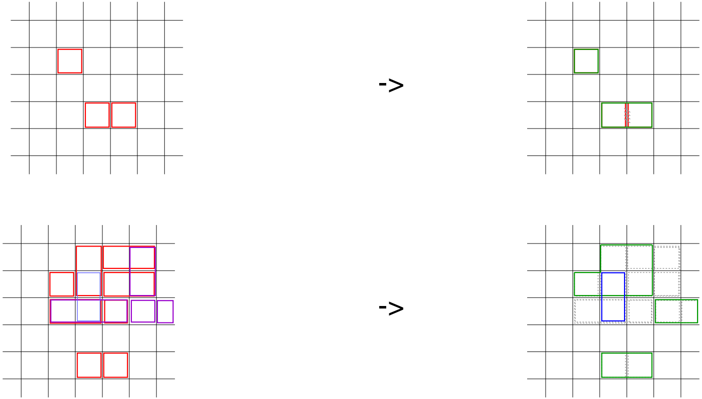

- Can you make a theorem out of this observation?

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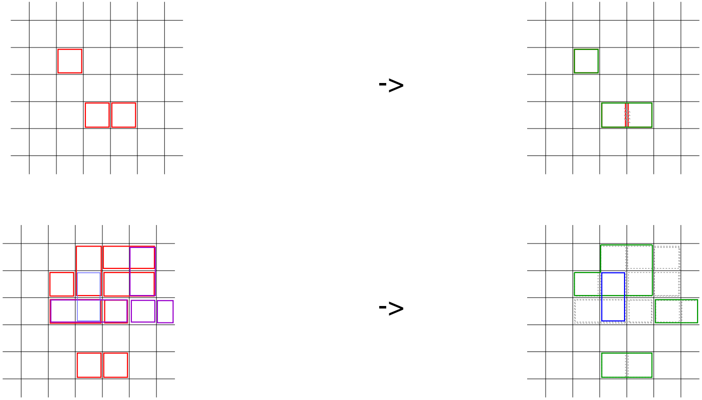

- Can you make a theorem out of this observation?
- Is it useful?

The cycle space of a finite graph

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Proposition

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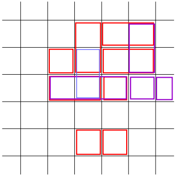

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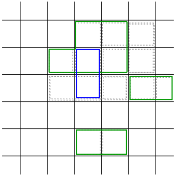
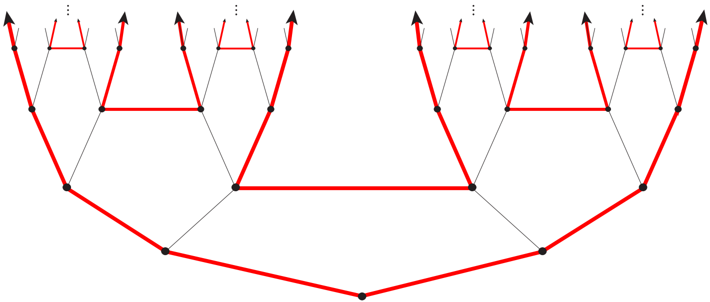


The wild circle

Circle: A homeomorphic image of S^1 in $|G|$.



the wild circle of Diestel & Kühn

Cycle decompositions for infinite graphs

Theorem (Diestel & Kühn)

Every element of the topological cycle space $\mathcal{C}(G)$ of a locally finite graph G can be written as a union of a set of edge-disjoint circles.

MacLane's Planarity Criterion

Theorem (MacLane '37)

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Theorem (Bruhn & Stein'05)

... verbatim generalisation for locally finite G

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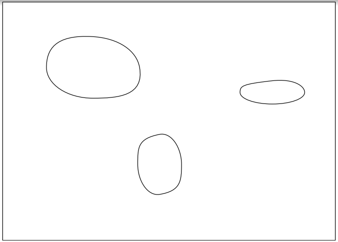

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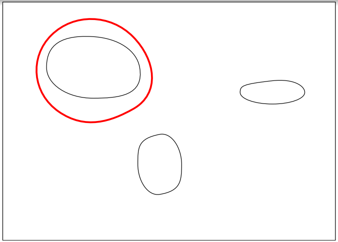

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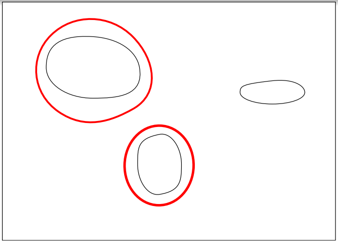

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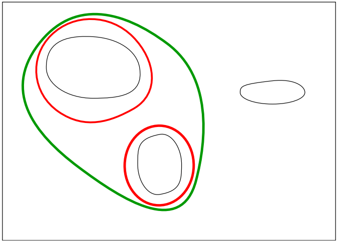

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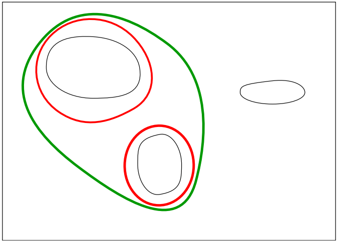

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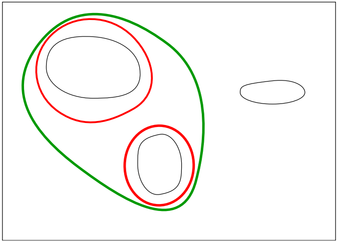

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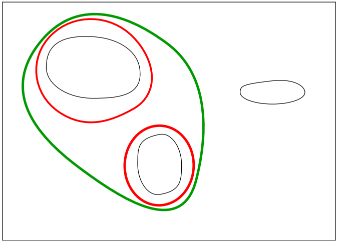

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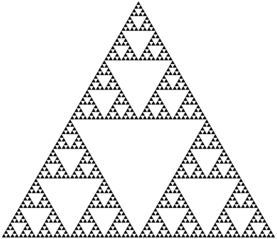

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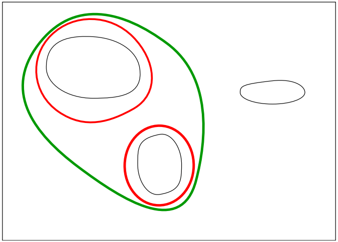

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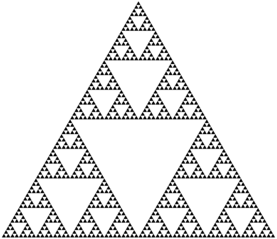

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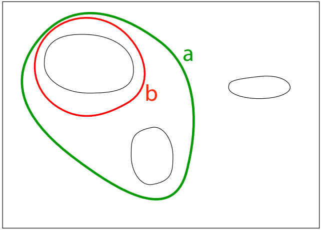

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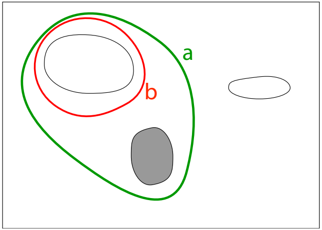

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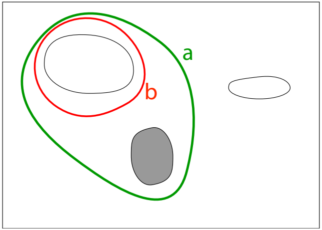

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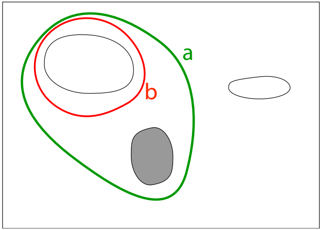

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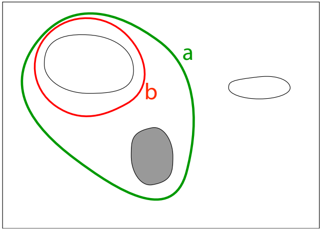

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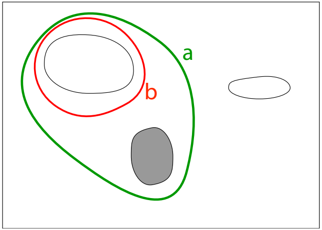

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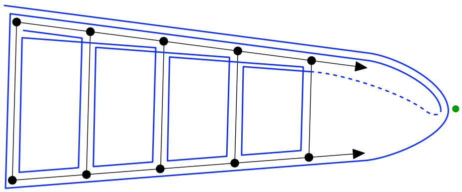
Let $H_1'(X) := H_1(X)/_{d=0}$ and let $\hat{H}_1(X)$ be its completeion.

An example



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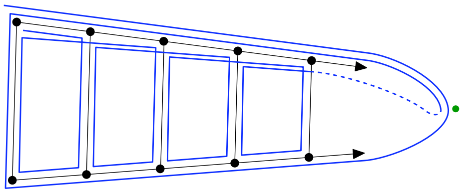

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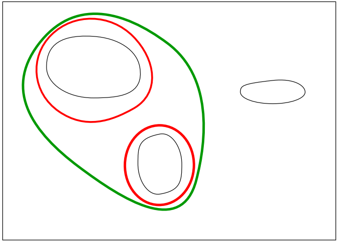
The Theorem



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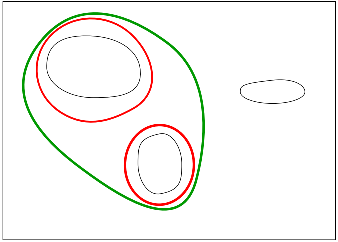

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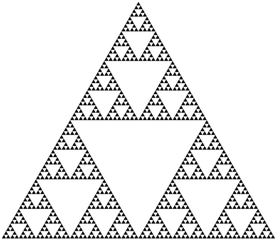
Theorem (G' 09)

For every compact metric space X and $C \in \hat{H}_1(X)$, there is a representative $(z_i)_{i \in \mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.

The Conjecture



Theorem (MacLane '37)

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Conjecture

Let X be a compact, 1–dimensional, locally connected, metrizable space that has no cut point. Then X is planar iff there is a simple set S of loops in X and a metric d inducing the topology of X so that the set $U := \{[\chi] \in \hat{H}_1(X) \mid \chi \in S\}$ spans $\hat{H}_1(X)$.

An intermediate result

Let $(\Gamma, +)$ be an abelian metrizable topological group, and suppose a function $\ell : \Gamma \to \mathbb{R}^+$ is given satisfying the following properties

- $\ell(a) = 0$ iff $a = 0$;
- $\ell(a + b) \leq \ell(a) + \ell(b)$ for every $a, b \in \Gamma$;
- if $b = \lim a_i$ then $\ell(b) \leq \liminf \ell(a_i)$;
- Some "isoperimetric inequality" holds: e.g. $d(a, 0) \leq U\ell^2(a)$ for some fixed U and for every $a \in \Gamma$.

Then every element of Γ is a (possibly infinite) sum of primitive elements.

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- Generalise to higher dimensions

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- Can you modify $\hat{H}_1$ to obtain a homology that is invariant under homotopy–equivalence?