From finite graphs to infinite; and beyond

Agelos Georgakopoulos

Technische Universität Graz
and
Mathematisches Seminar
Universität Hamburg

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Things that go wrong in infinite graphs

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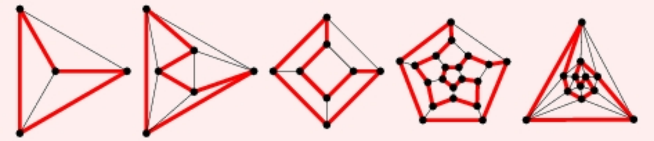
- Hamilton cycle theorems
- Extremal graph theory
- Cycle space theorems
- many others ...

Hamilton cycles

Hamilton cycle: A cycle containing all vertices.

Some examples:



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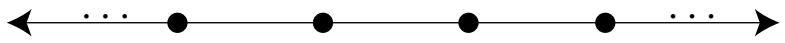
$\Rightarrow$ need more general notions

Spanning Double-Rays

Classical approach to 'save' Hamilton cycle theorems:
accept double-rays as infinite cycles



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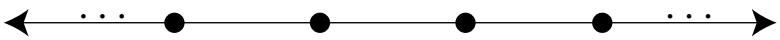

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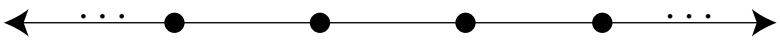


This approach only extends finite theorems in very restricted cases:

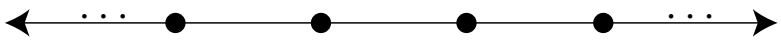
Theorem (Tutte ’56)

Every finite 4-connected planar graph has a Hamilton cycle

4-connected := you can remove any 3 vertices and the graph remains connected

Spanning Double-Rays

Classical approach: accept double-rays as infinite cycles



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Theorem (Yu '05)

Every locally finite 4-connected planar graph has a spanning double ray ...

Spanning Double-Rays

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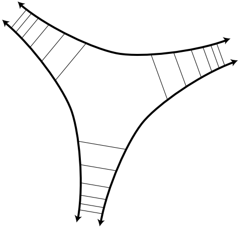
This approach only extends finite theorems in very restricted cases:

Theorem (Yu '05)

Every locally finite 4-connected planar graph has a spanning double ray ... unless *it is 3-divisible.*

Compactifying by Points at Infinity

A 3-divisible graph



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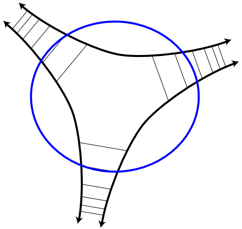
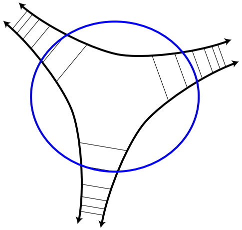


Compactifying by Points at Infinity

A 3-divisible graph
can have no spanning double ray



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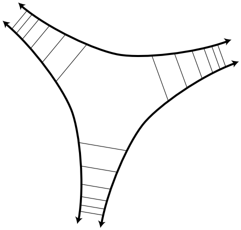

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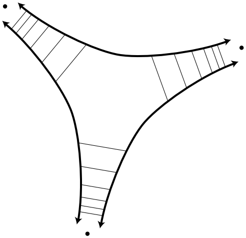

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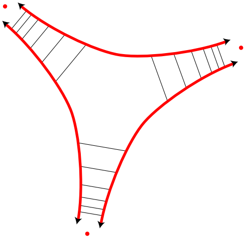

... but a Hamilton cycle?

Ends

end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them

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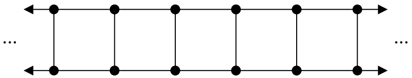

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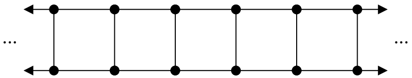

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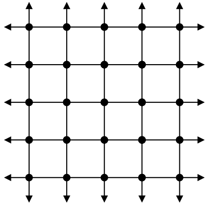

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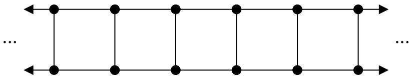

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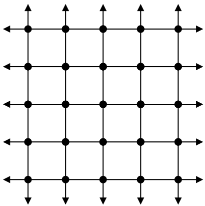

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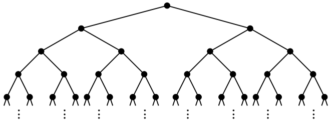
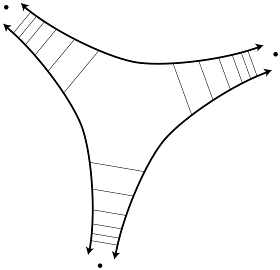


uncountably many ends

The End Compactification



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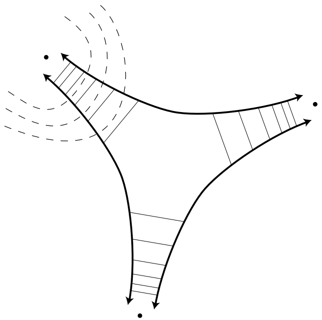

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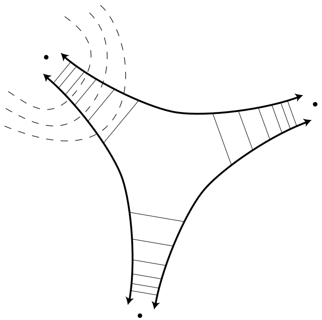
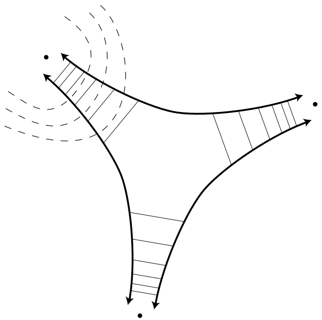


Every ray converges to its end

The End Compactification

$|G|$



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(Equivalent) definition of $|G|$

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Theorem (G '06)

If $\sum_{e \in E(G)} \ell(e) < \infty$ then $|G|_\ell$ is homeomorphic to $|G|$.

Infinite Cycles

Circle:
A homeomorphic image of S^1 in $|G|$.

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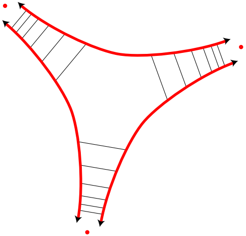

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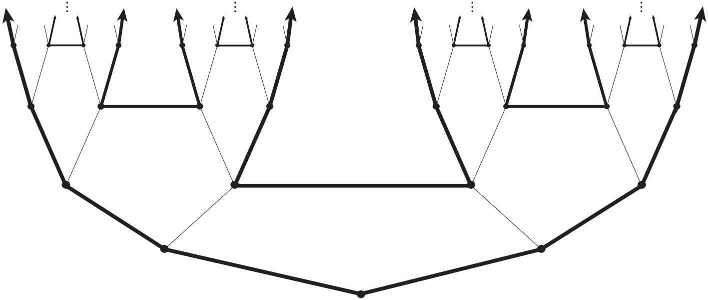

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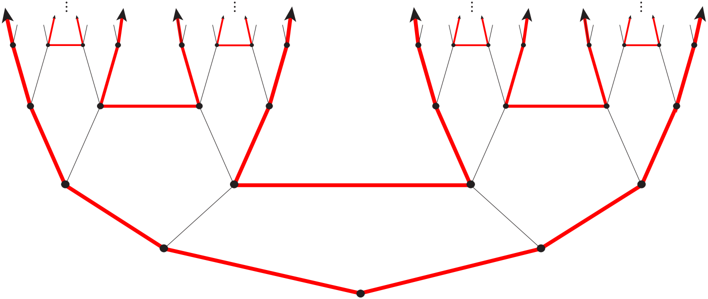

the wild circle of Diestel & Kühn

Fleischner's Theorem

Theorem (Fleischner '74)

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The square of a finite 2-connected graph has a Hamilton cycle

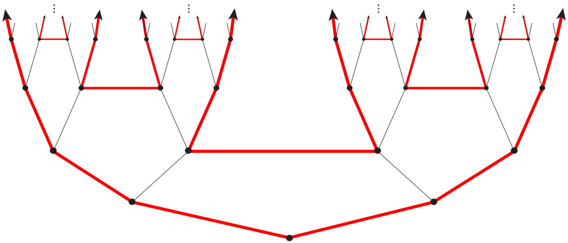
Theorem (Thomassen '78)

The square of a locally finite 2-connected 1-ended graph has a Hamilton circle.

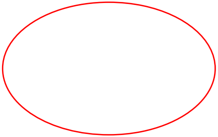
The Theorem

Theorem (G '06)

The square of any locally finite 2-connected graph has a Hamilton circle



Proof?



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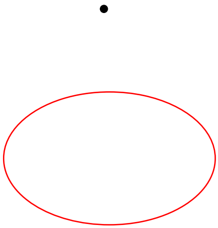

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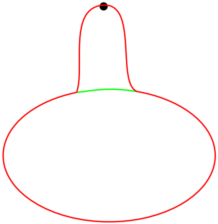

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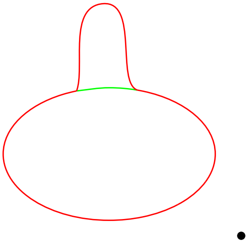

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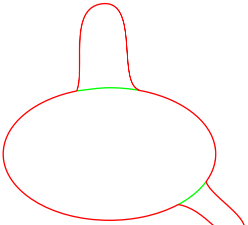

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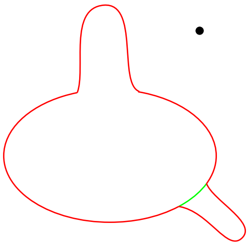

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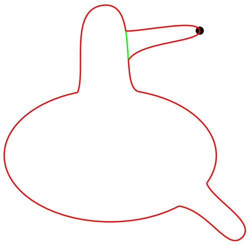

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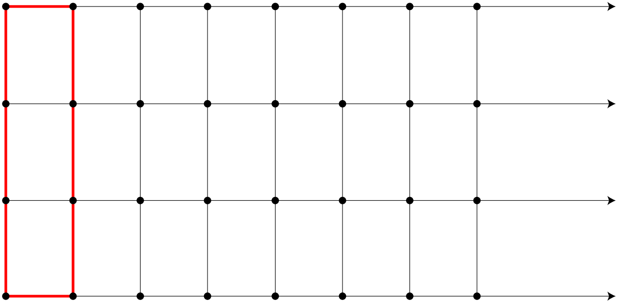

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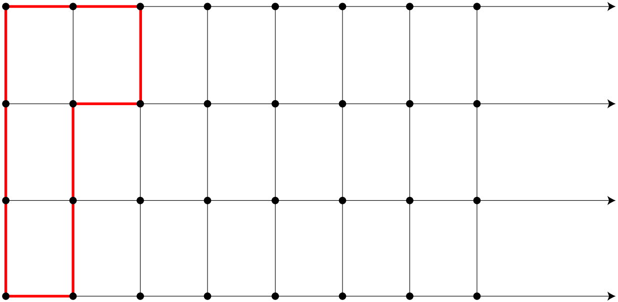

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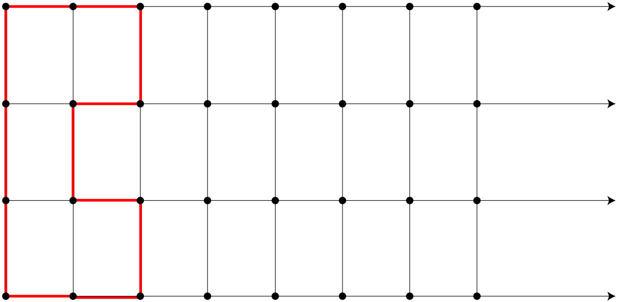

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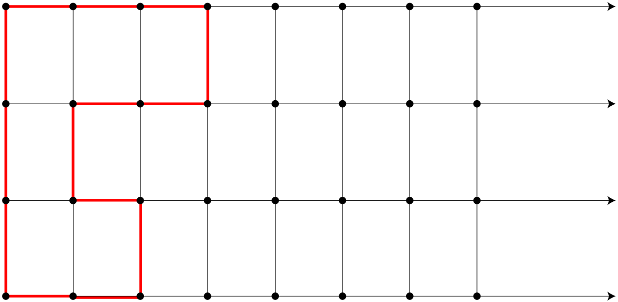

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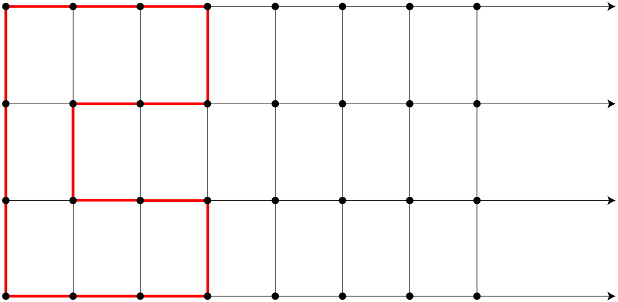

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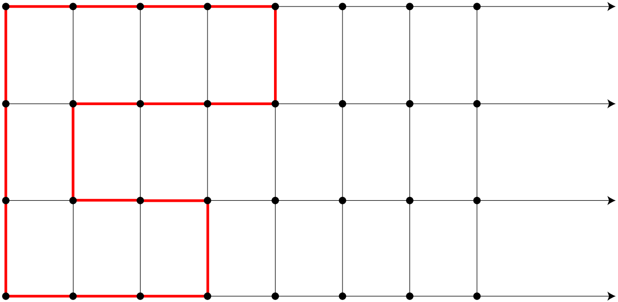

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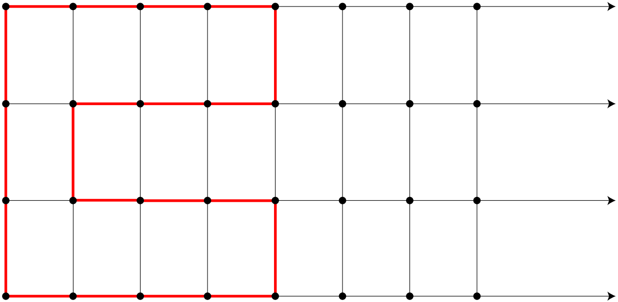

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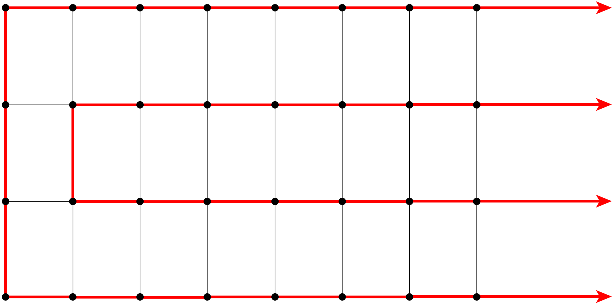

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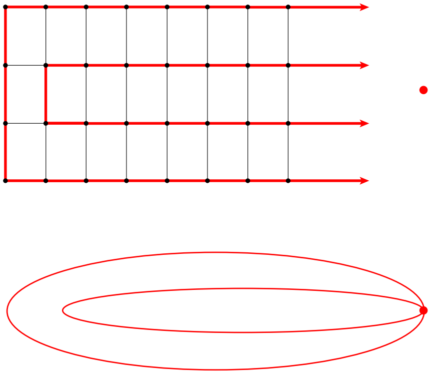

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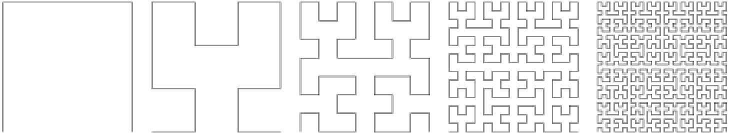
Hilbert's space filling curve:

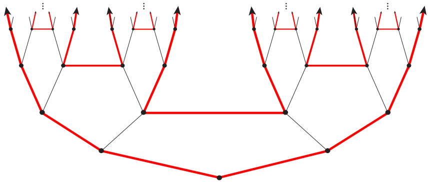


a sequence of injective curves with a non-injective limit

The Theorem

Theorem (G '06)

The square of any locally finite 2-connected graph has a Hamilton circle



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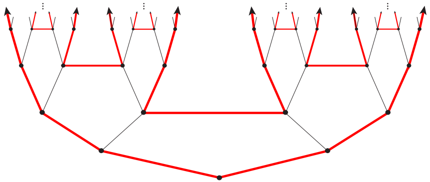

Corollary

Cayley graphs are "morally" hamiltonian.

Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser '59)

Does every finite connected Cayley graph have a Hamilton cycle?

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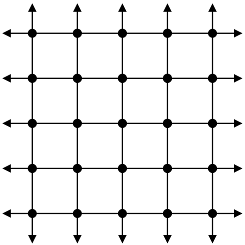

Hamiltonicity in Cayley graphs

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Problem

Characterise the locally finite Cayley graphs that admit Hamilton circles.

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Cycle Space

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- A vector space over $\mathbb{Z}_2$
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- Allows infinite sums (whenever well-defined).

The topological Cycle Space

Known facts:

- A connected graph has an Euler tour iff every edge-cut is even (Euler)
- G is planar iff $\mathcal{C}(G)$ has a simple generating set (MacLane)
- The relator-cycles of a Cayley graph G generate $\mathcal{C}(G)$.

Generalisations:

Bruhn & Stein

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Bruhn & G

MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

simple: no edge appears in more than two generators.

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Theorem (Bruhn & Stein'05)

... verbatim generalisation for locally finite G

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Theorem (Diestel & Sprüssel' 09)

$\mathcal{C}(G)$ coincides with the first Čech homology group of $|G|$ but not with its first singular homology group.

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Problem

Can we use concepts from homology to generalise theorems from graphs to other topological spaces?

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Theorem (Bruhn & G '06)

Yes if E is countable, no otherwise

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Is $\langle \mathcal{N} \rangle = \langle\langle \mathcal{N} \rangle\rangle$?

Some linear algebra

Let R be a ring and E any set
Consider the module R^E
If $\mathcal{T} \subseteq R^E$ is thin, then we can define $\sum \mathcal{T}$
thin: for every coordinate $e \in E$ there are only finitely many elements $N \in \mathcal{T}$ with $N(e) \neq 0$.

Problem

Is $\langle \mathcal{N} \rangle = \langle\langle \mathcal{N} \rangle\rangle$*?*

Theorem (Bruhn & G '06)

Yes if $\mathcal{N}$ *is thin and* R *is a field or a finite ring, no otherwise*

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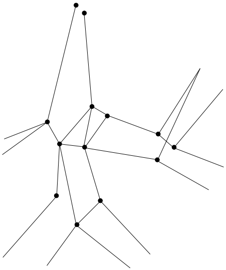
Theorem (G '08)

If $\sum_{e \in E} r(e) < \infty$ then there is a unique *non-elusive electrical flow of finite energy.*

energy := $\sum_{e \in E} i^2(e) r(e)$.

Assume, there are two 'good' flows f, g and consider

$$z := f - g$$



Finding wild circles by a limit construction

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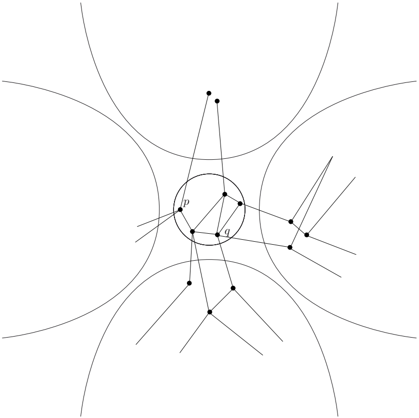

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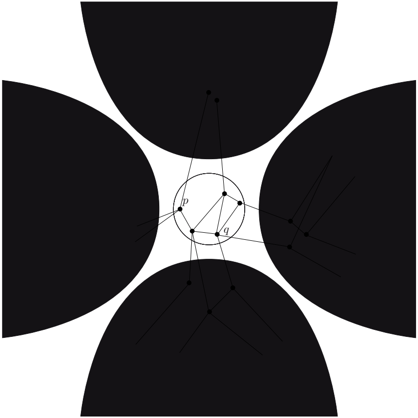

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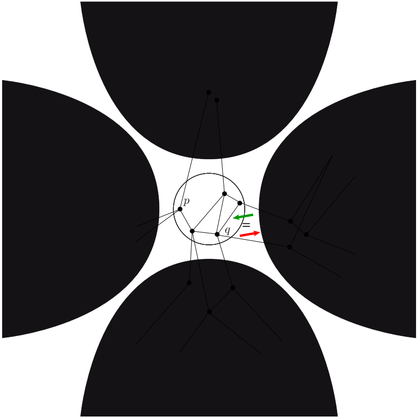

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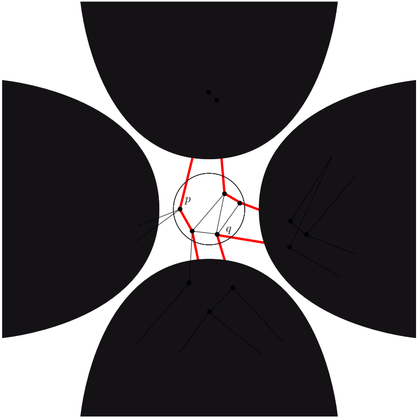

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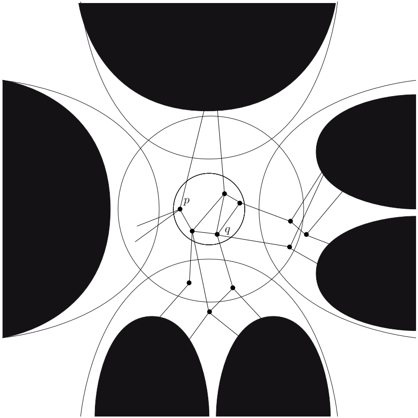

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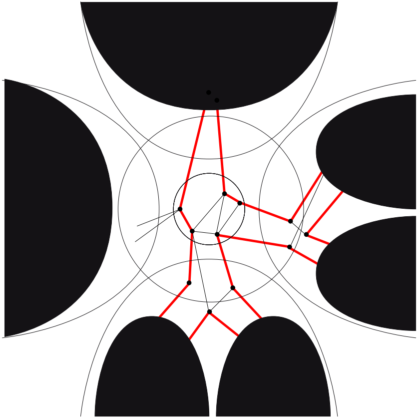

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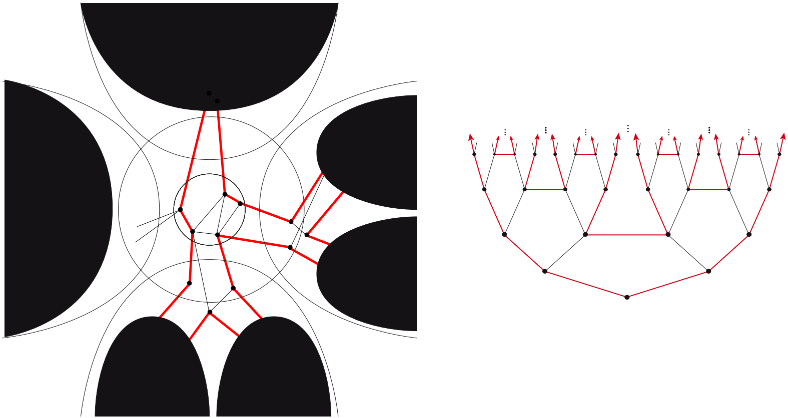

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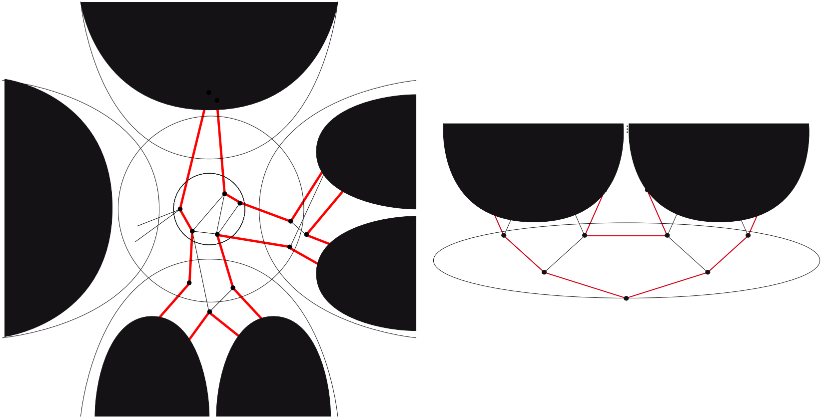

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