Square Tilings and the Poisson Boundary

Agelos Georgakopoulos

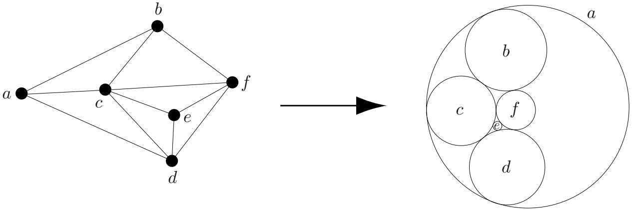
THE UNIVERSITY OF WARWICK

Cambridge, 17/10/13

The circle packing theorem

The Koebe-Andreev-Thurston circle packing theorem

For every finite planar graph G, there is a circle packing in the plane (or S^2) with nerve G.
The packing is unique (up to Möbius transformations) if G is a triangulation of S^2.



The Riemann mapping theorem

Theorem (Riemann? '1851, Carathéodory 1912)

For every simply connected open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$, there is a bijective conformal map from Ω onto the open unit disk.

The Riemann mapping theorem

Theorem (Riemann? '1851, Carathéodory 1912)

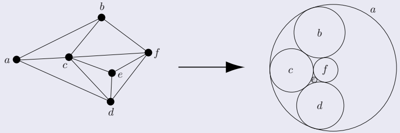
For every simply connected open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$, there is a bijective conformal map from Ω onto the open unit disk.

Theorem (Koebe 1908)

For every open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$ with finitely many boundary components, there is a bijective conformal map from Ω onto a circle domain.

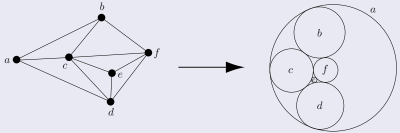
Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem



Circle Packing <=> Conformal map

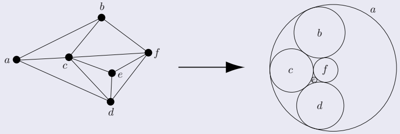
The Koebe-Andreev-Thurston circle packing theorem



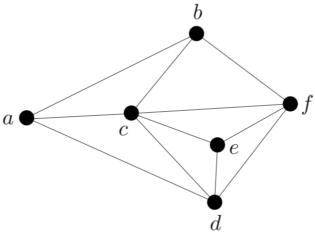
Circle Packing <= Conformal map

Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem

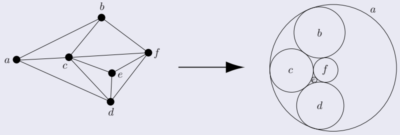


Circle Packing <= Conformal map

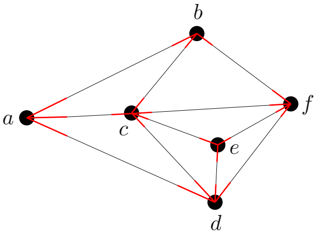


Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem

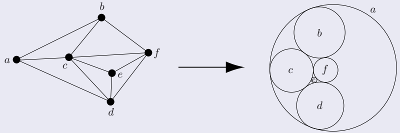


Circle Packing <= Conformal map

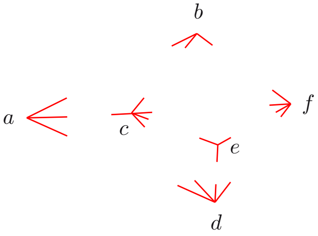


Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem

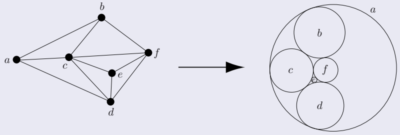


Circle Packing <= Conformal map

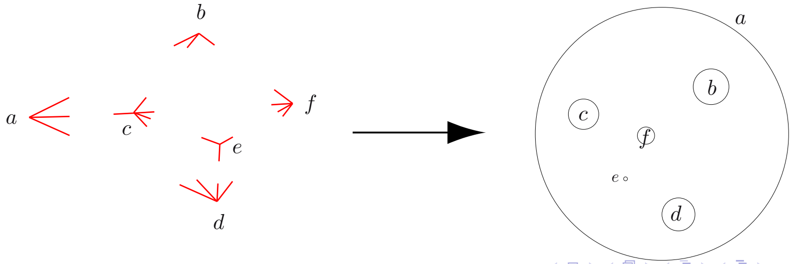


Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem

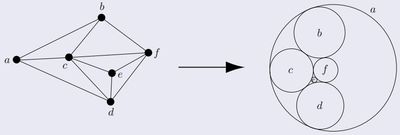


Circle Packing <= Conformal map

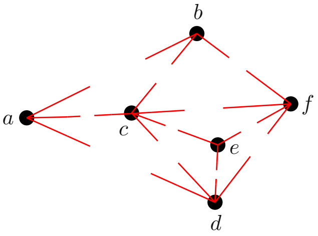


Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem

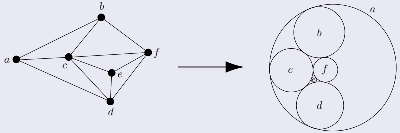


Circle Packing <= Conformal map

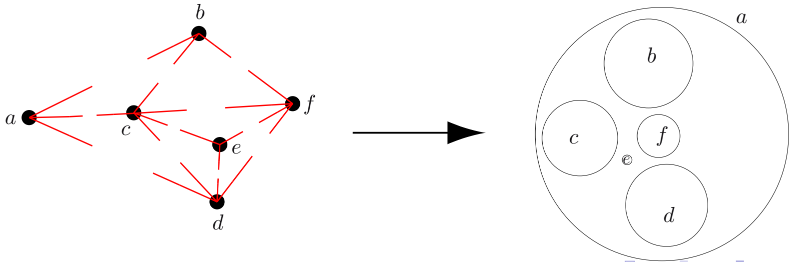


Circle Packing <=> Conformal map

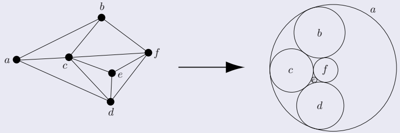
The Koebe-Andreev-Thurston circle packing theorem



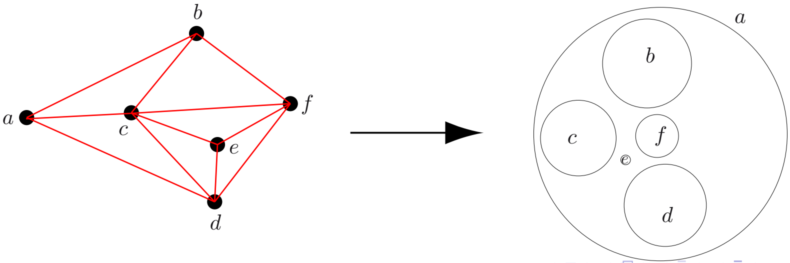
Circle Packing <= Conformal map



The Koebe-Andreev-Thurston circle packing theorem

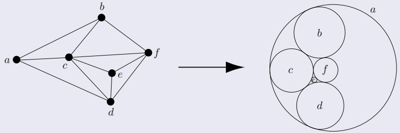


Circle Packing <= Conformal map

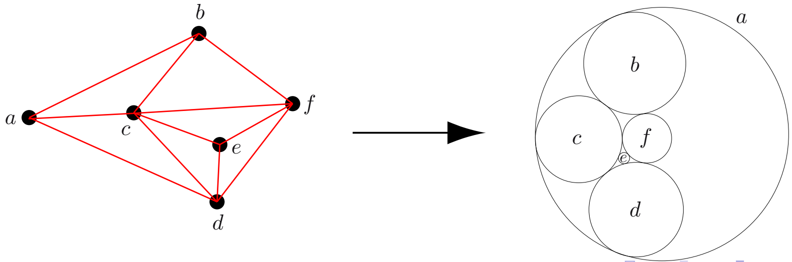


Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem

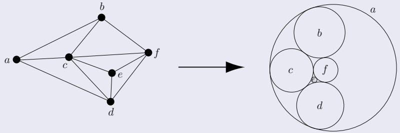


Circle Packing <= Conformal map



Circle Packing <=> Conformal map

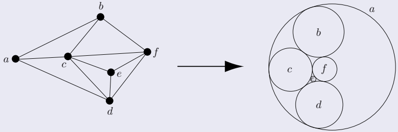
The Koebe-Andreev-Thurston circle packing theorem



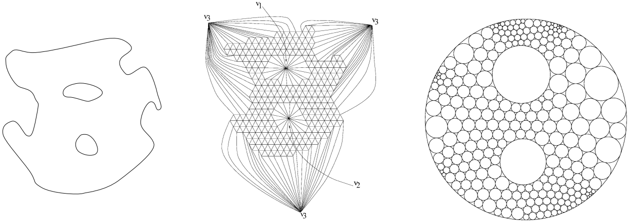
Circle Packing => Conformal map

Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem

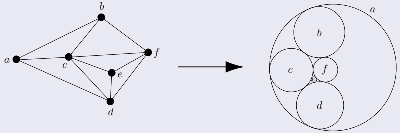


Circle Packing => Conformal map

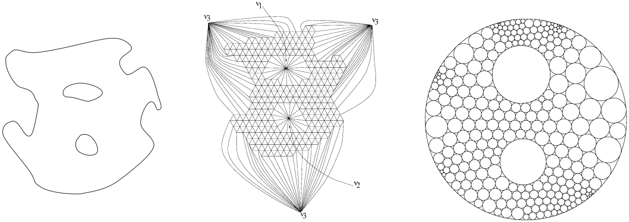


Circle Packing <=> Conformal map

The Koebe-Andreev-Thurston circle packing theorem



Circle Packing => Conformal map

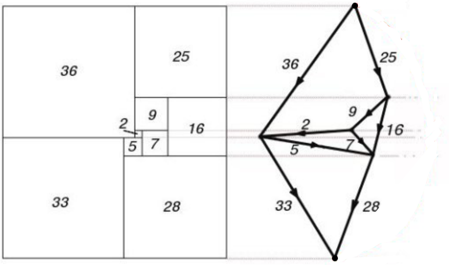


[*S. Rohde: "Oded Schramm: From Circle Packing to SLE", '10*]

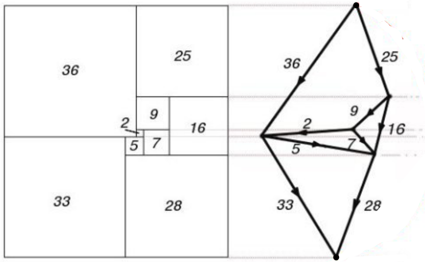
Square Tilings

Theorem (Brooks, Smith, Stone & Tutte '40)

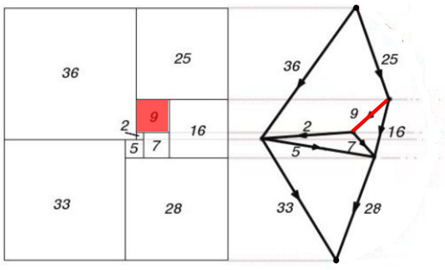
... for every finite planar graph G, there is a square tiling with incidence graph G ...



Properties of square tilings

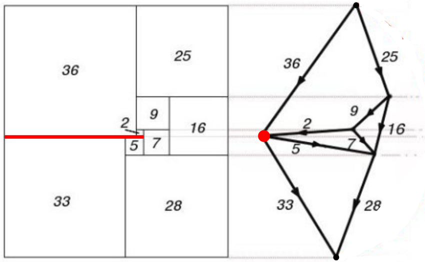


Properties of square tilings



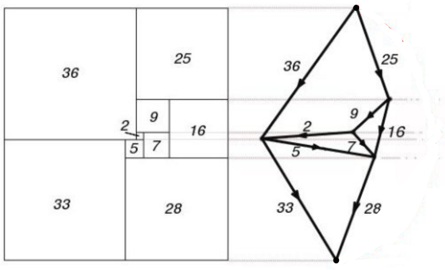
- every edge is mapped to a square;

Properties of square tilings



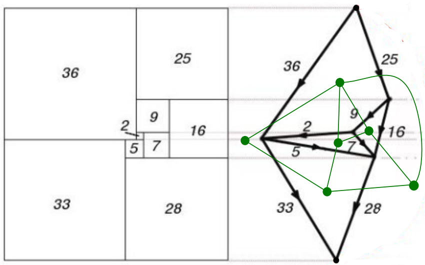
- every edge is mapped to a square;
- vertices correspond to horizontal segments tangent with their edges;

Properties of square tilings

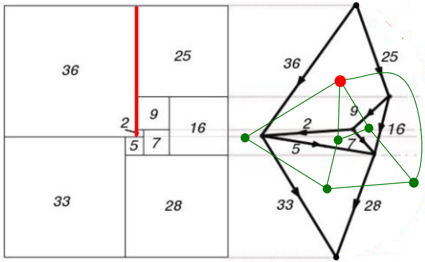


- every edge is mapped to a square;
- vertices correspond to horizontal segments tangent with their edges;
- there is no overlap of squares, and no 'empty' space left;

Properties of square tilings

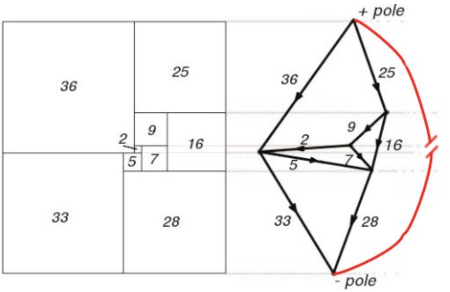


- every edge is mapped to a square;
- vertices correspond to horizontal segments tangent with their edges;
- there is no overlap of squares, and no ‘empty’ space left;
- the square tiling of the dual of G can be obtained from that of G by a 90° rotation.



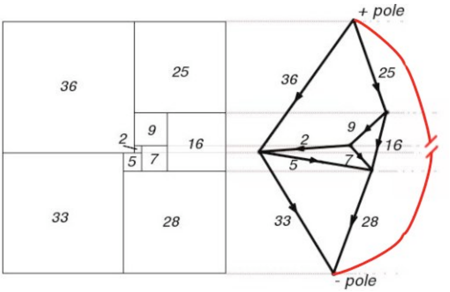
- every edge is mapped to a square;
- vertices correspond to horizontal segments tangent with their edges;
- there is no overlap of squares, and no 'empty' space left;
- the square tiling of the dual of G can be obtained from that of G by a 90° rotation.

The construction of square tilings



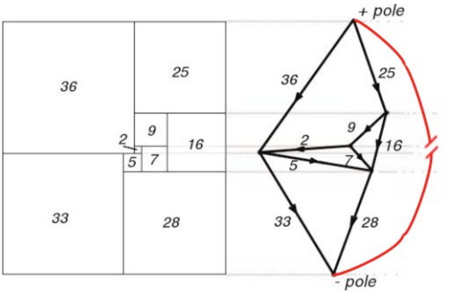
- Think of the graph as an electrical network;

The construction of square tilings



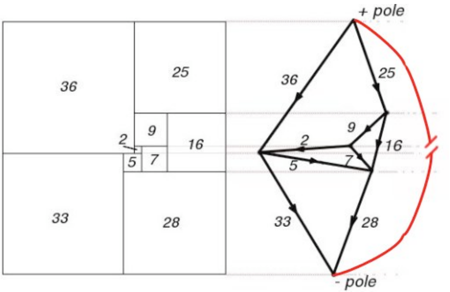
- Think of the graph as an electrical network;
- impose an electrical current from p to q;

The construction of square tilings



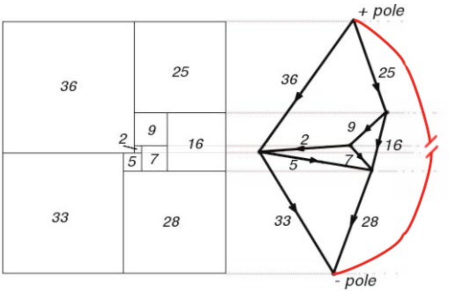
- Think of the graph as an electrical network;
- impose an electrical current from p to q;
- let the square corresponding to edge e have side length the flow $i(e)$;

The construction of square tilings



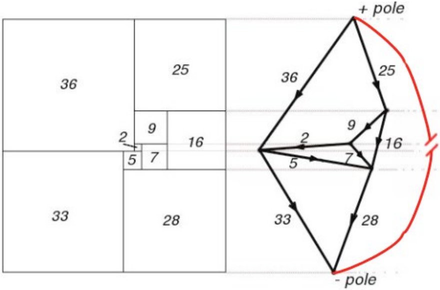
- Think of the graph as an electrical network;
- impose an electrical current from p to q;
- let the square corresponding to edge e have side length the flow $i(e)$;
- place each vertex x at height equal to the potential $h(x)$;

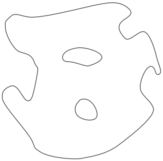
The construction of square tilings

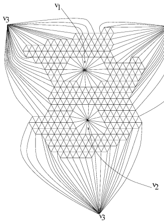


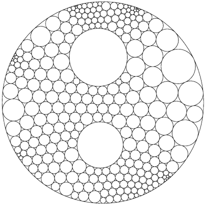
- Think of the graph as an electrical network;
- impose an electrical current from p to q;
- let the square corresponding to edge e have side length the flow $i(e)$;
- place each vertex x at height equal to the potential $h(x)$;
- use a duality argument to determine the width coordinates.

The construction of square tilings

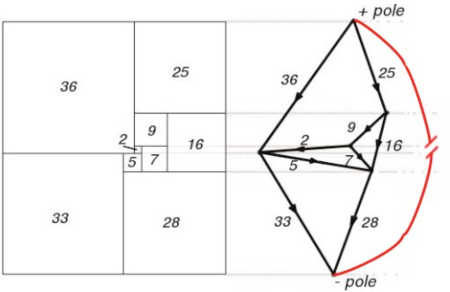






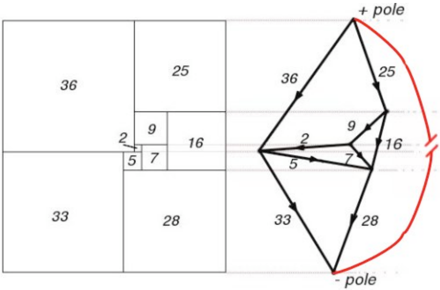


The construction of square tilings



- Think of the graph as an electrical network;
- impose an electrical current from p to q;
- let the square corresponding to edge e have side length the flow $i(e)$;
- place each vertex x at height equal to the potential $h(x)$;
- use a duality argument to determine the width coordinates.

The construction of square tilings

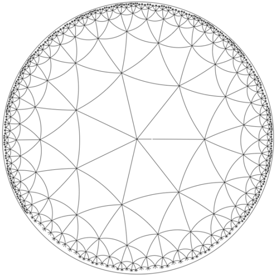


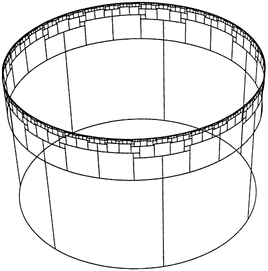
[*J. W. Cannon, W. J. Floyd, and W. R. Parry: "Squaring rectangles: The finite Riemann mapping theorem."*]

The square tilings of Benjamini & Schramm

Theorem (Benjamini & Schramm '96)

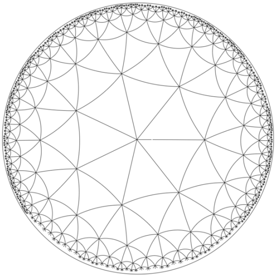
Every transient (infinite) graph G of bounded degree that has a uniquely absorbing embedding in the plane admits a square tiling.

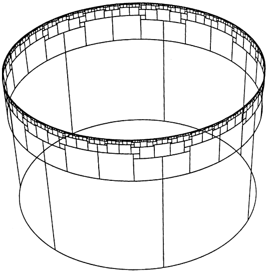


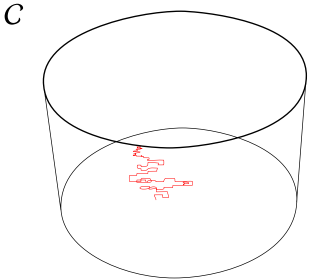


Theorem (Benjamini & Schramm '96)

Every transient (infinite) graph G of bounded degree that has a uniquely absorbing embedding in the plane admits a square tiling. Moreover, random walk on G converges a. s. to a point in $\mathcal{C}$.







The Poisson integral representation formula

The classical Poisson formula

$$h(z) = \int_0^1 \hat{h}(\theta) P(z,\theta) d\theta$$

where $P(z,\theta) := \frac{1-|z|^2}{|e^{2\pi i\theta}-z|^2}$,

recovers every continuous harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

The Poisson integral representation formula

The classical Poisson formula

$$h(z) = \int_0^1 \hat{h}(\theta) P(z,\theta) d\theta = \int_0^1 \hat{h}(\theta) d\nu_z(\theta)$$

where $P(z,\theta) := \frac{1-|z|^2}{|e^{2\pi i\theta}-z|^2}$,

recovers every continuous harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

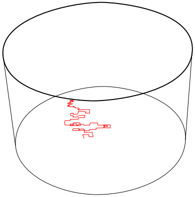
The Poisson integral representation formula

The classical Poisson formula

$$h(z) = \int_0^1 \hat{h}(\theta) P(z,\theta) d\theta = \int_0^1 \hat{h}(\theta) d\nu_z(\theta)$$

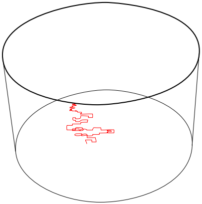
where $P(z,\theta) := \frac{1-|z|^2}{|e^{2\pi i\theta}-z|^2}$,

recovers every continuous harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.



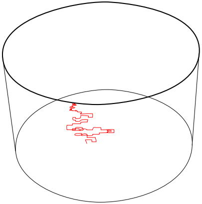
The boundary of the square tiling coincides with the Poisson boundary

Can the bounded harmonic functions on a plane graph G be expressed as a Poisson-like integral using $\mathcal{C}$?



The boundary of the square tiling coincides with the Poisson boundary

Can the bounded harmonic functions on a plane graph G be expressed as a Poisson-like integral using $\mathcal{C}$?

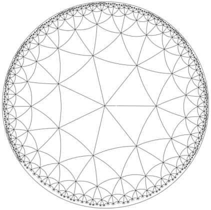


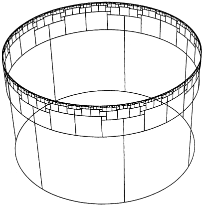
A function $h : V(G) \to \mathbb{R}$,
is **harmonic**, if $h(x) = \sum_{y \sim x} h(y)/d(x)$.

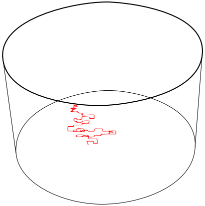
The boundary of the square tiling coincides with the Poisson boundary

Question (Benjamini & Schramm '96)

Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?



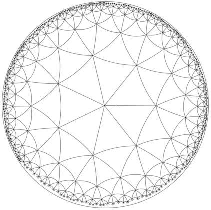


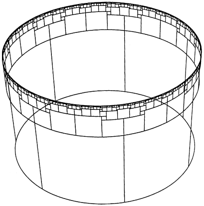


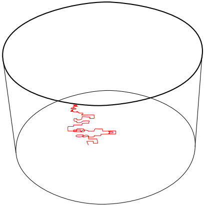
The boundary of the square tiling coincides with the Poisson boundary

Question (Benjamini & Schramm '96)

Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?







Theorem (G '12)

Yes!

The Poisson-Furstenberg boundary

The Poisson boundary of an (infinite) graph G consists of

- a measurable space $(\mathcal{P}_G, \Sigma)$, and

The Poisson-Furstenberg boundary

The Poisson boundary of an (infinite) graph G consists of

- a measurable space $(\mathcal{P}_G, \Sigma)$, and
- a family of probability measures $\{\nu_z, z \in V_G\}$,

such that

The Poisson-Furstenberg boundary

The Poisson boundary of an (infinite) graph G consists of
- a measurable space $(\mathcal{P}_G, \Sigma)$, and
- a family of probability measures $\{\nu_z, z \in V_G\}$,

such that

- every bounded harmonic function h can be obtained by

$$h(z) = \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$$

The Poisson-Furstenberg boundary

The Poisson boundary of an (infinite) graph G consists of

- a measurable space $(\mathcal{P}_G, \Sigma)$, and
- a family of probability measures $\{\nu_z, z \in V_G\}$,

such that

- every bounded harmonic function h can be obtained by

$$h(z) = \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$$

- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;

The Poisson-Furstenberg boundary

The Poisson boundary of an (infinite) graph G consists of

- a measurable space $(\mathcal{P}_G, \Sigma)$, and
- a family of probability measures $\{\nu_z, z \in V_G\}$,

such that

- every bounded harmonic function h can be obtained by

$$h(z) = \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$$

- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;
- conversely, for every $\hat{h} \in L^\infty(\mathcal{P}_G)$ the function $z \mapsto \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$ is bounded and harmonic.

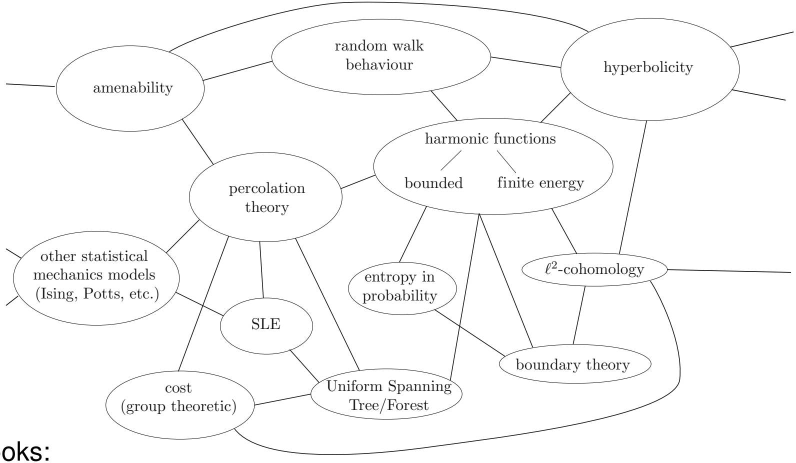
i.e. there is Poisson-like formula establishing an isometry between the Banach spaces $H^\infty(G)$ and $L^\infty(\mathcal{P}_G)$.

The Poisson-Furstenberg boundary

Selected work on the Poisson boundary

- Introduced by Furstenberg to study semi-simple Lie groups [*Annals of Math. '63*]
- Kaimanovich & Vershik give a general criterion using the entropy of random walk [*Annals of Probability '83*]
- Kaimanovich identifies the Poisson boundary of hyperbolic groups, and gives general criteria [*Annals of Math. '00*]

The context



Textbooks:

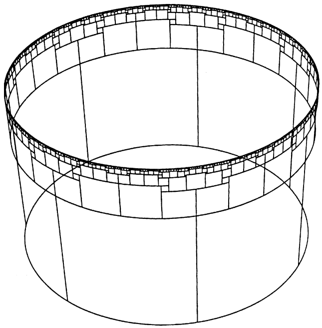
[*Woess: Random Walks on Infinite Graphs and Groups*]

[*Lyons & Peres: Probability on Trees and Networks*]

The theorem

Theorem (G '12)

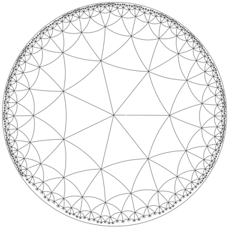
For every bounded degree graph admitting a square tiling, the Poisson boundary coincides with C.

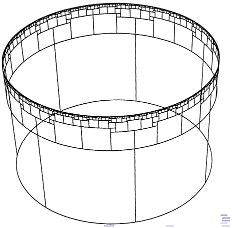


Probabilistic interpretation of the tiling

Lemma (G '12)

Let C be a 'horizontal' circle in the tiling T of G, and let B the set of points of G at which C 'dissects' T. Then the widths of the points of B in T coincide with the probability distribution of the first visit to B by brownian motion on G starting at o.

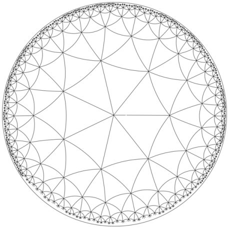


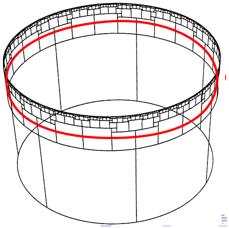


Probabilistic interpretation of the tiling

Lemma (G '12)

Let C be a 'horizontal' circle in the tiling T of G, and let B the set of points of G at which C 'dissects' T. Then the widths of the points of B in T coincide with the probability distribution of the first visit to B by brownian motion on G starting at o.

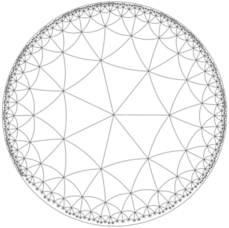


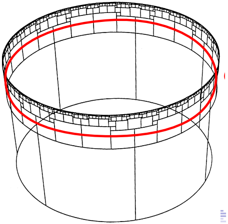


Probabilistic interpretation of the tiling

Lemma

For every 'meridian' M in T, the probability that brownian motion on G starting at o will 'cross' M clockwise equals the probability to cross M counter-clockwise.

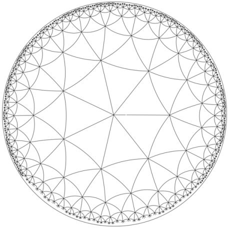


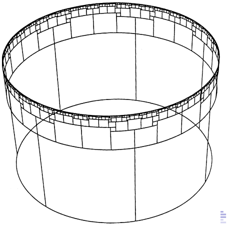


Probabilistic interpretation of the tiling

Lemma

For every ‘meridian’ M in T, the probability that brownian motion on G starting at o will ‘cross’ M clockwise equals the probability to cross M counter-clockwise.





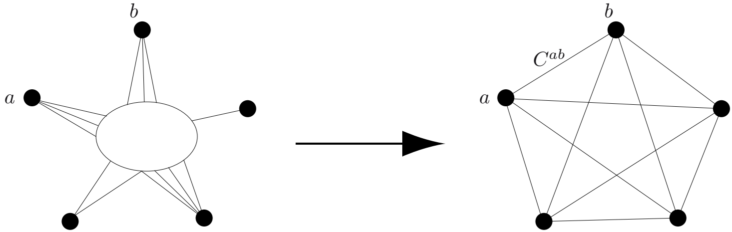
Here come some
'geometric' random graphs

Electrical Network Reduction

Theorem

Let N be an electrical network and B its set of external nodes. Then there is an equivalent network with vertex set B in which each edge (a, b) *has conductance*

$$C_{eff}(a, b) = d(a)\mathbb{P}_a(b).$$



The Effective Conductance Measure

For any infinite graph G, we construct a measure space $\mathcal{C} = \mathcal{C}(G)$ that allows expressing the energy of harmonic functions as an integral on the boundary:

The Effective Conductance Measure

For any infinite graph G, we construct a measure space $\mathcal{C} = \mathcal{C}(G)$ that allows expressing the energy of harmonic functions as an integral on the boundary:

Theorem (G & V. Kaimanovich '13+)

For every transient locally finite network N there is a measure $\mathcal{C}$ on $\mathcal{P}^2$ such that for every harmonic function u with boundary function $\widehat{u}$,

$$E(u) = \int_{\mathcal{P}^2} \left(\widehat{u}(\eta) - \widehat{u}(\zeta)\right)^2 d\mathcal{C}(\eta, \zeta).$$

The Effective Conductance Measure

For any infinite graph G, we construct a measure space $\mathcal{C} = \mathcal{C}(G)$ that allows expressing the energy of harmonic functions as an integral on the boundary:

Theorem (G & V. Kaimanovich '13+)

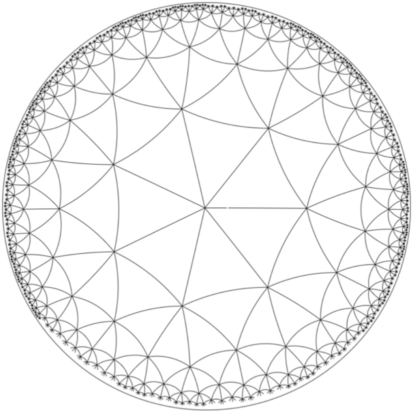
For every transient locally finite network N there is a measure $\mathcal{C}$ on $\mathcal{P}^2$ such that for every harmonic function u with boundary function $\widehat{u}$,

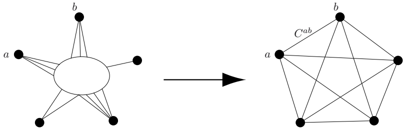
$$E(u) = \int_{\mathcal{P}^2} \left(\widehat{u}(\eta) - \widehat{u}(\zeta)\right)^2 d\mathcal{C}(\eta, \zeta).$$

Energy $E(u) := \sum_{x \sim y} (u(x) - u(y))^2$

The Effective Conductance Measure

$$E(u) = \int_{\mathcal{P}^2} \left(\widehat{u}(\eta) - \widehat{u}(\zeta)\right)^2 dC(\eta, \zeta).$$





Summary