

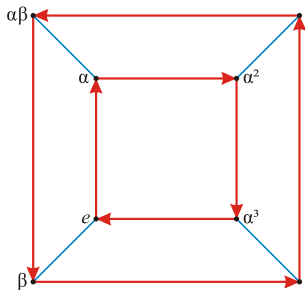
The planar cubic Cayley graphs

Agelos Georgakopoulos

U of Ottawa / TU Graz

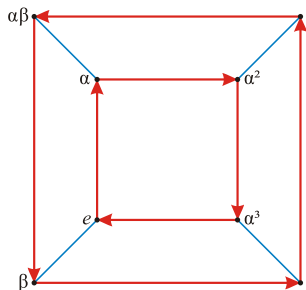
Dartmouth College, 12.05.11

Cayley graphs



$$\langle \alpha, \beta \mid \beta^2, \alpha^4, (\alpha\beta)^2 \rangle$$

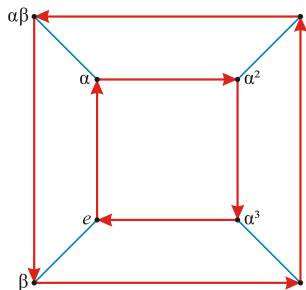
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Cayley graphs

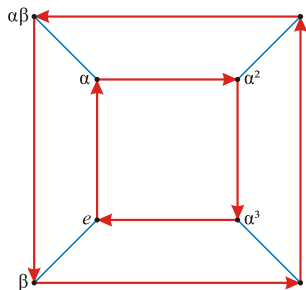


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$$g \xrightarrow{s} gs$$

Sabidussi's Theorem

Theorem (Sabidussi's Theorem)

A properly edge-coloured digraph is a Cayley graph iff for every $x, y \in V(G)$ there is a colour-preserving automorphism mapping x to y .

properly edge-coloured := no vertex has two incoming or two outgoing edges with the same colour

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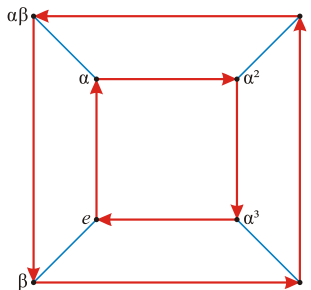
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Characterisation of the finite planar groups

Theorem (Maschke 1886)

Every finite planar group is a group of isometries of S^2 .



planar group := a group having at least 1 planar Cayley graph.

The Cayley complex

Let $\Gamma = \langle a, b, c, \dots \mid R_1, R_2 \dots \rangle$ be a group presentation.

Define the corresponding **simplified Cayley complex**

$CC \langle a, b, c, \dots \mid R_1, R_2 \dots \rangle$ by:

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Given a planar Cayley graph, can you find a presentation in which the relators induce precisely the face boundaries?

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Given a planar Cayley graph, can you find a presentation in which the relators induce precisely the face boundaries?

Yes!

Proving Maschke's Theorem

Given a finite plane Cayley graph G , consider the following group presentation:

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Theorem (Whitney '32)

Let G be a 3-connected plane graph. Then every automorphism of G preserves face-boundaries.

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Since $\Gamma(G)$ acts on X , we have:

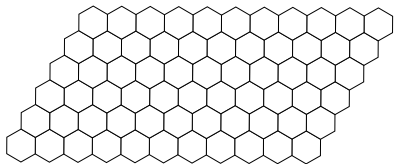
Theorem (Maschke 1886)

Every finite planar group is a group of homeomorphisms of S^2 .

The 1-ended planar groups

Theorem ((classic) Macbeath, Wilkie, ...)

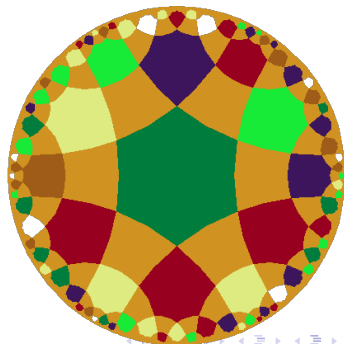
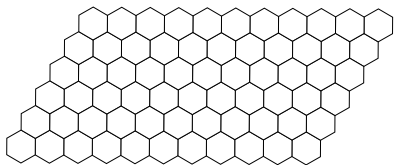
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Planar groups and fundamental groups of surfaces

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What about the other ones?

Theorem (G '10)

A group has a planar simplified Cayley complex if and only if it has a VAP-free Cayley graph.

What about the non VAP-free ones?

Open Problems:

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How can you split a planar Cayley graph with > 1 ends into simpler Cayley graphs?

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Classification of the cubic planar Cayley graphs

Theorem (G '10)

*Let G be a planar cubic Cayley graph. Then G is colour-isomorphic to precisely one element of **the list**.*

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*Let G be a planar cubic Cayley graph. Then G is colour-isomorphic to precisely one element of **the list**.*

Conversely, for every element of the list and any choice of parameters, the corresponding Cayley graph is planar.

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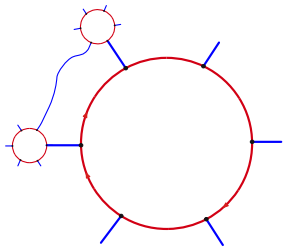
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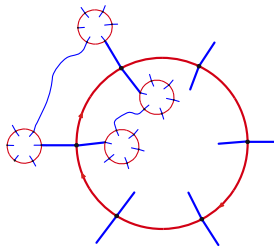
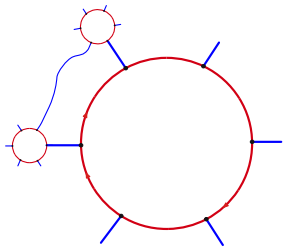
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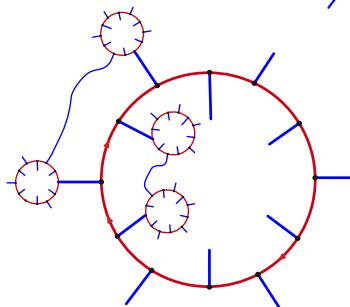
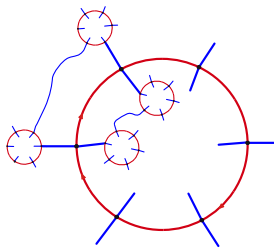
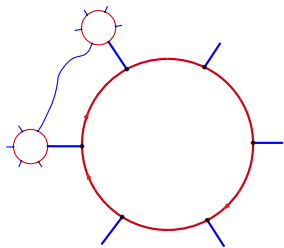
Examples



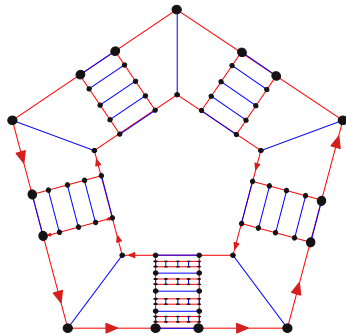
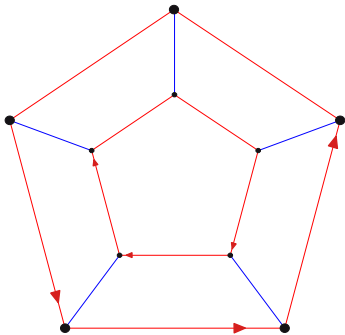
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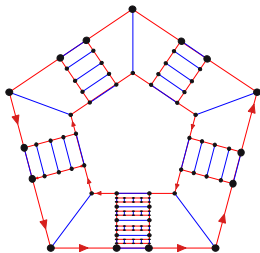
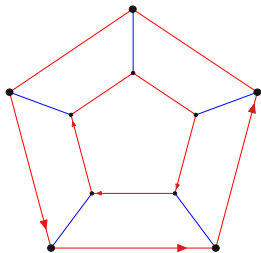
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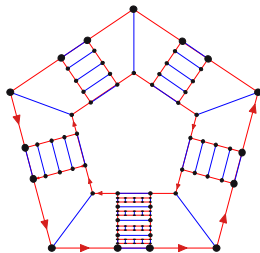
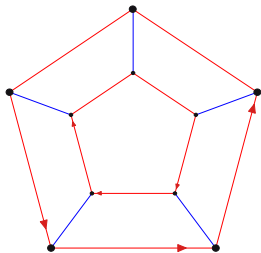
Examples



Corollary (G '10)

Every planar cubic Cayley graph has an almost planar Cayley complex.

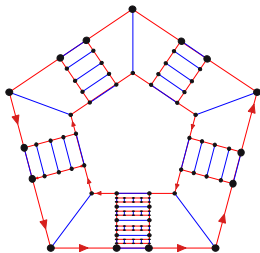
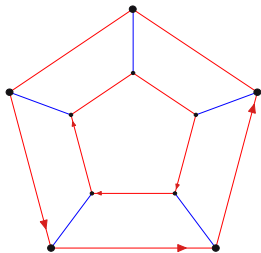
Examples



Corollary (G & Hamann '11)

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Examples



Corollary (G & Hamann '11)

*Every planar cubic Cayley graph has
an almost planar Cayley complex...
maybe*

Cayley graphs without finite face boundaries

Conjecture (Bonnington
& Watkins)

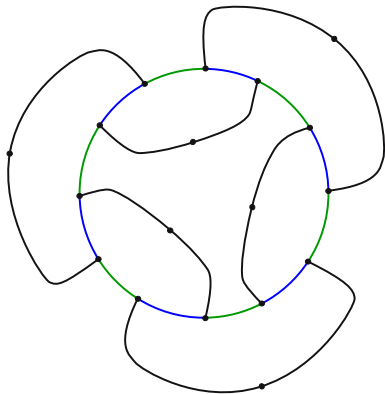
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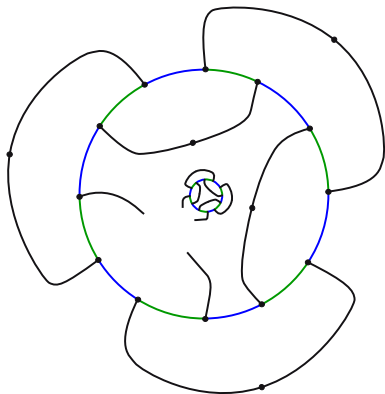
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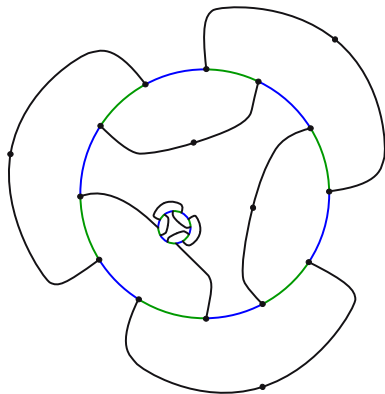
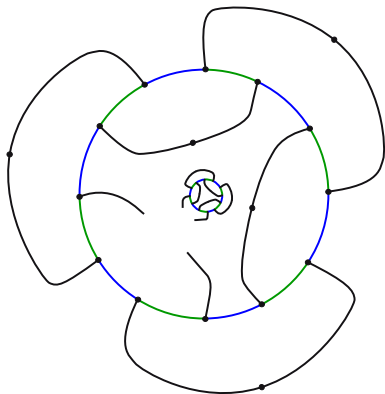
Cayley graphs without finite face boundaries



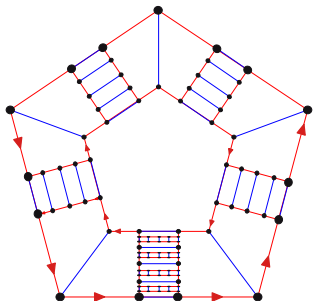
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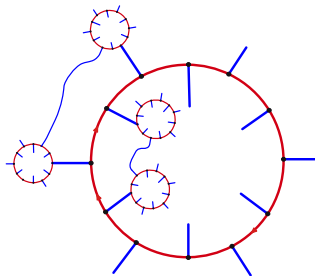
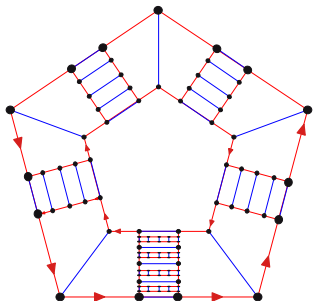
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Spot the societies!



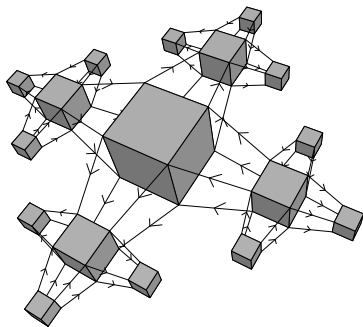
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Stallings' Theorem

Theorem (Stallings '71)

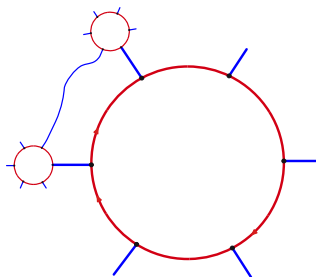
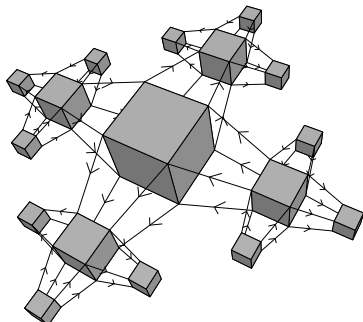
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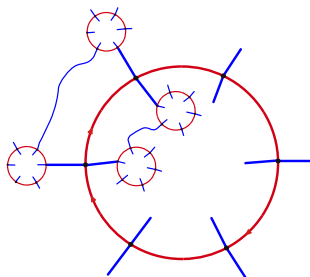
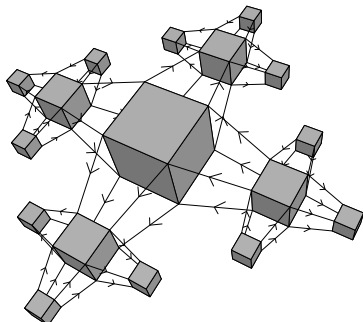
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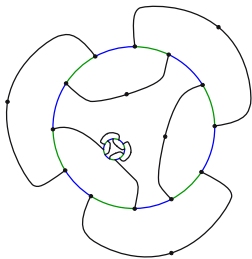
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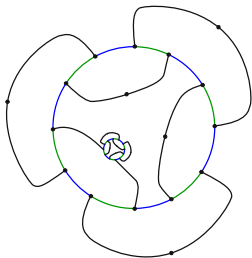
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Group splittings by topological minors



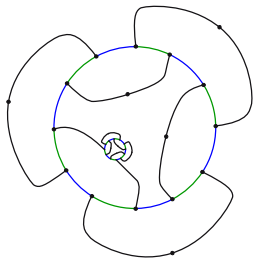
Group splittings by topological minors



Conjecture

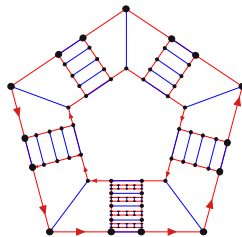
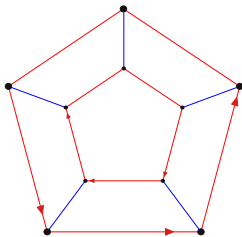
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Group splittings by topological minors

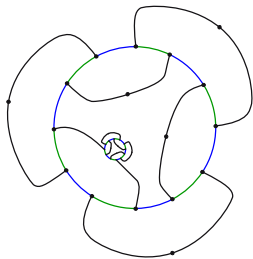


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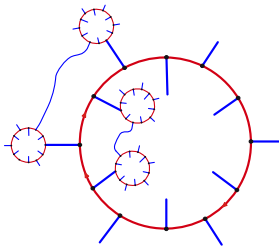


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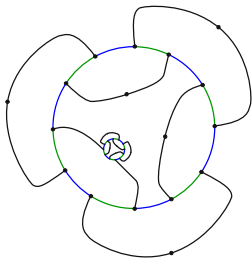


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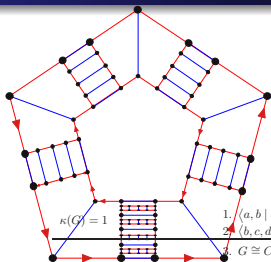
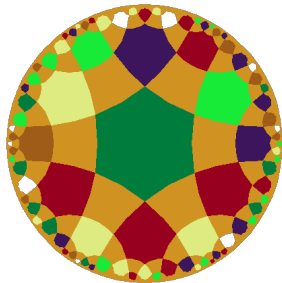
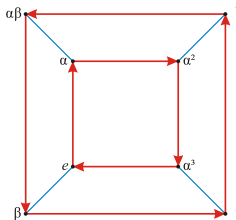
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Corollary (G '10)

True for planar cubic Cayley graphs.

Summary



$\kappa(G) = 2$

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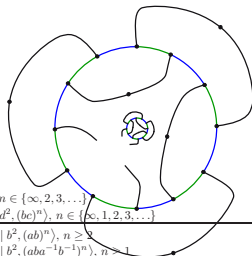
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with two generators

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$\kappa(G) = 3$,

G is multi-ended,
with two generators



1. $\langle a, b \mid b^2, a^n \rangle, n \in \{\infty, 2, 3, \dots\}$
2. $\langle b, c, d \mid b^2, c^2, d^2, (bc)^n \rangle, n \in \{\infty, 1, 2, 3, \dots\}$
3. $G \cong \text{Cay} \langle a, b \mid b^2, (ab)^n \rangle, n \geq 2$
4. $G \cong \text{Cay} \langle a, b \mid b^2, (aba^{-1}b^{-1})^n \rangle, n \geq 1$
5. $G \cong \text{Cay} \langle a, b \mid b^2, a^4, (a^2b)^2 \rangle$
6. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bcd)^m \rangle, m \geq 2$
7. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (cbcd)^m \rangle, n, m \geq 2$
8. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bd)^m \rangle, n, m \geq 2$
9. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (b(cb)^n d)^m \rangle, n \geq 1, m \geq 2$
10. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bcdb)^m \rangle, m \geq 1$
11. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^n, cd \rangle, n \geq 1$
12. $G \cong \text{Cay} \langle a, b \mid b^2, a^n, (ab)^m \rangle, n \geq 3, m \geq 2$
13. $G \cong \text{Cay} \langle a, b \mid b^2, a^n, (aba^{-1}b)^m \rangle, n \geq 3, m \geq 1$
14. $G \cong \text{Cay} \langle a, b \mid b^2, (a^2b)^m \rangle, m \geq 1$
15. $G \cong \text{Cay} \langle a, b \mid b^2, (a^2ba^{-2}b)^m \rangle, m \geq 1$
16. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bcd)^n \rangle, n \geq 1$
17. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (cbcd)^m \rangle, n \geq 1$
18. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bdcd)^m \rangle, n \geq 2, m \geq 1$
19. $G \cong \text{Cay} \langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (cd)^m, (db)^p \rangle, n, m, p \geq 2$
20. $G \cong \text{Cay} \langle a, b \mid b^2, (a^2b)^m; a^{2n} \rangle, n \geq 3, m \geq 2$
21. $G \cong \text{Cay} \langle a, b \mid b^2, (a^2ba^{-2}b)^m; a^{2n} \rangle, n \geq 3, m \geq 1$
22. $G \cong \text{Cay} \langle a, b \mid b^2, a^2ba^{-2}b; (baba^{-1})^n \rangle, n \geq 2$
23. $G \cong \text{Cay} \langle a, b \mid b^2, (a^2ba^{-2}b)^m; (baba^{-1})^n \rangle, n, m, p \geq 2$
24. $G \cong \text{Cay} \langle a, b \mid b^2, (a^2b)^2; (\overline{ab})^{2m} \rangle, m \geq 2$
25. $G \cong \text{Cay} \langle a, b \mid b^2, (a^2b)^2; (\overline{ab})^{2m} \rangle, m \geq 2$