The Dirichlet problem in a network of finite total resistance

Agelos Georgakopoulos

Technische Universität Graz
and
Mathematisches Seminar
Universität Hamburg

Dubrovnik, 2.6.09

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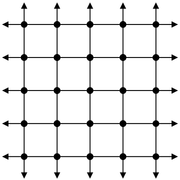

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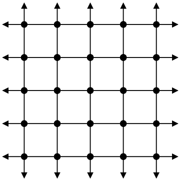

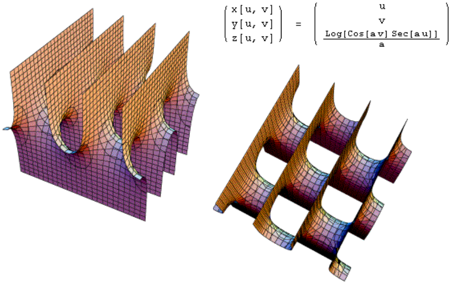

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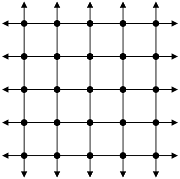

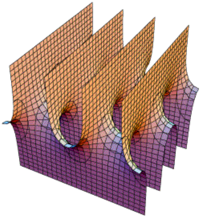

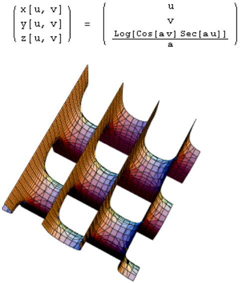

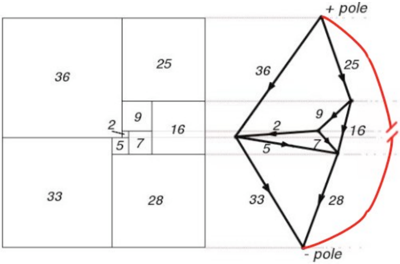

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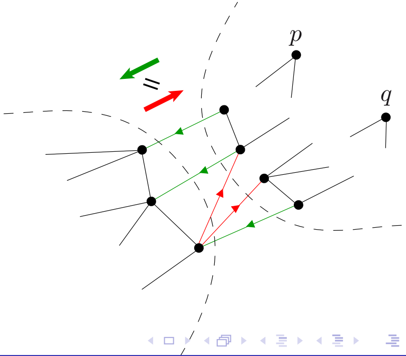

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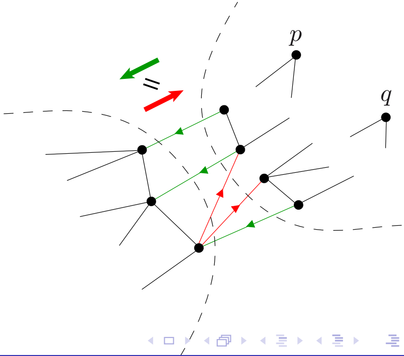
Non-elusive flow:
The net flow along any finite cut must be zero:



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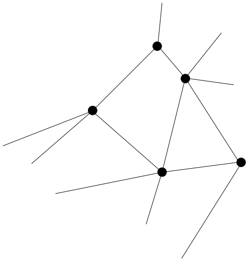

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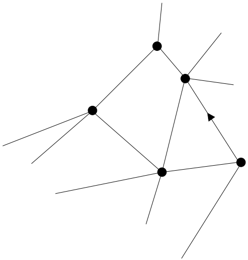

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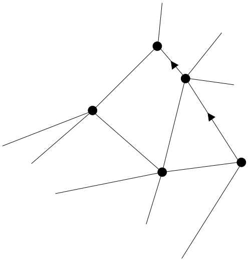

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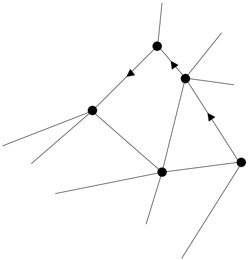

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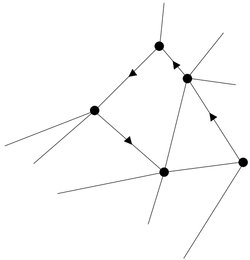

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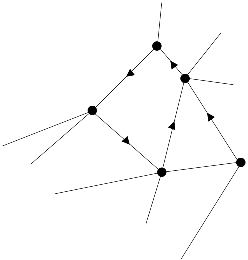

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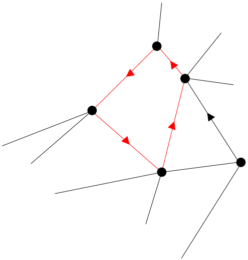

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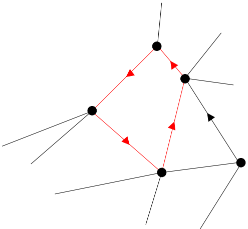

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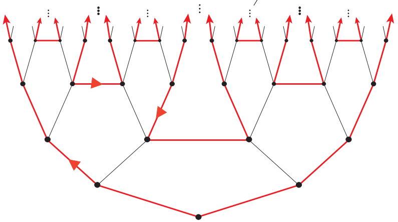

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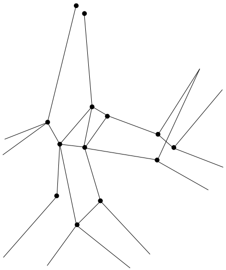

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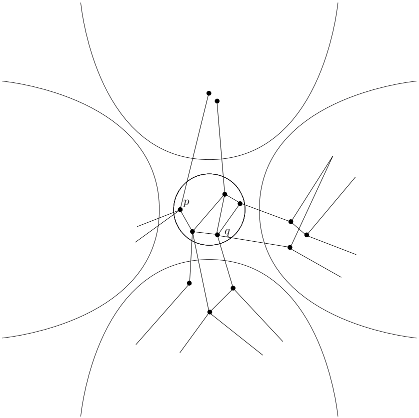

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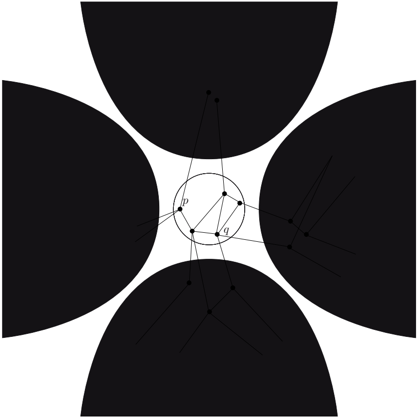

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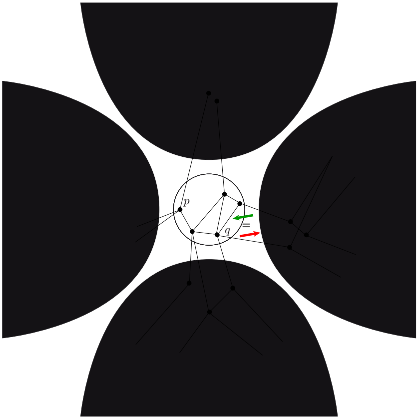

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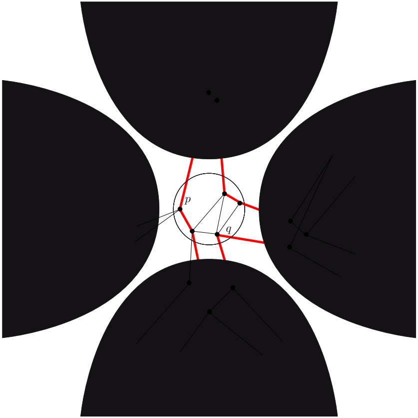

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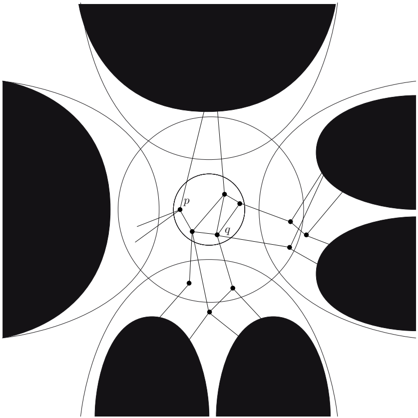

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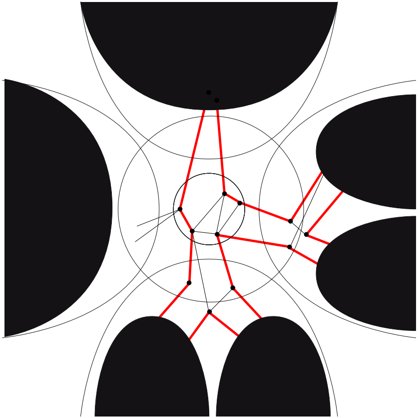

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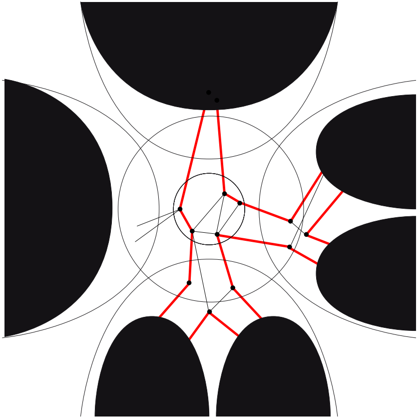

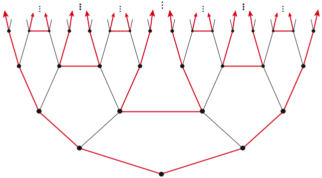

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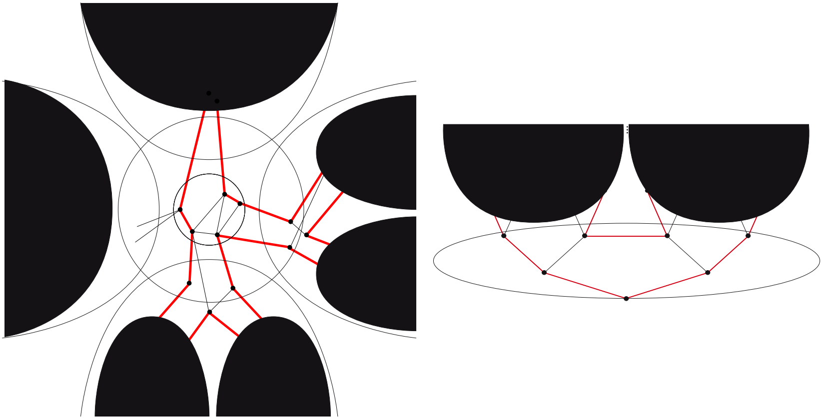

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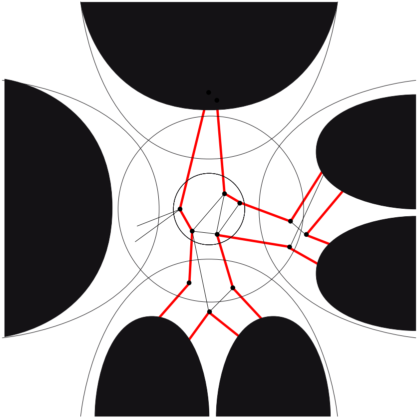

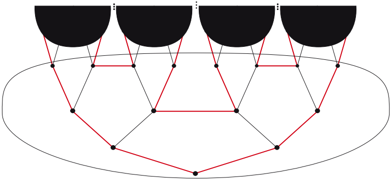

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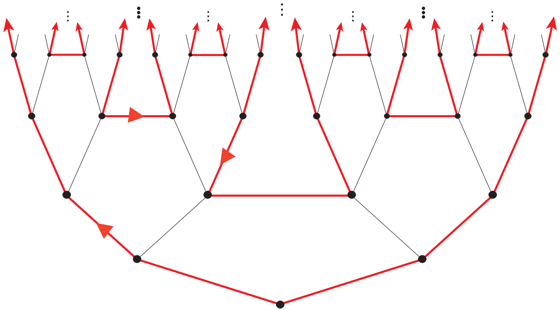

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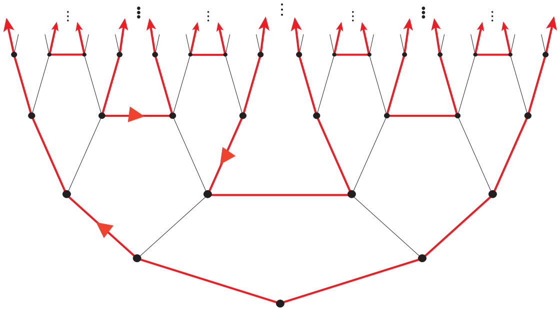

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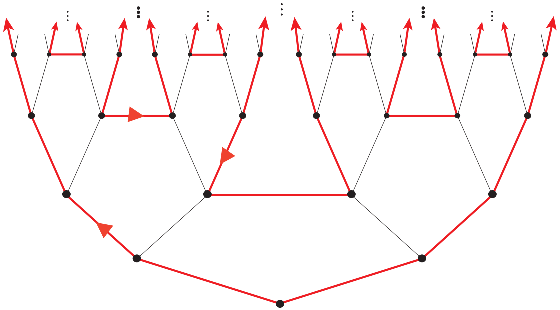

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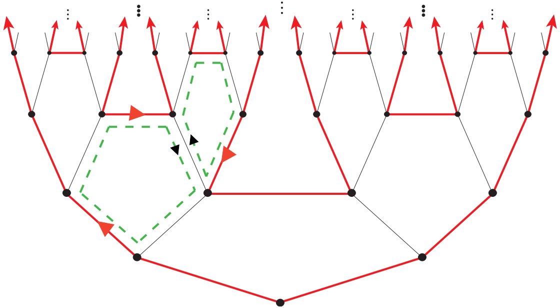

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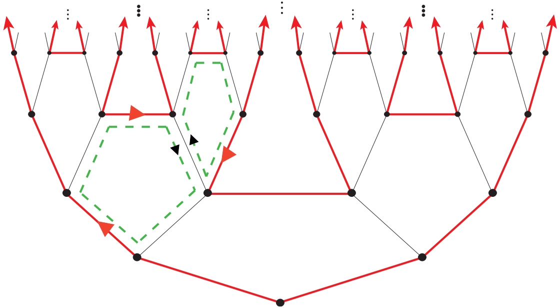

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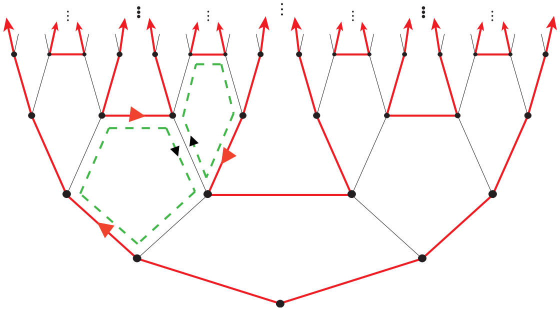

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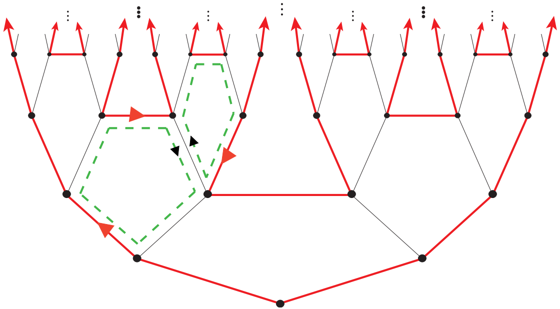

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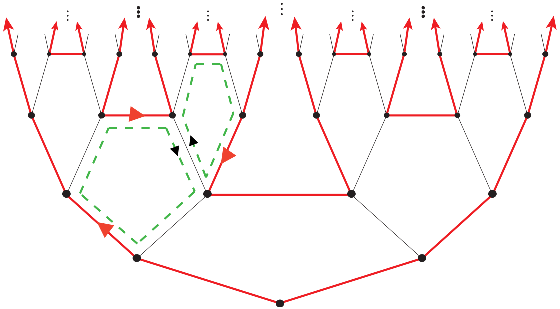

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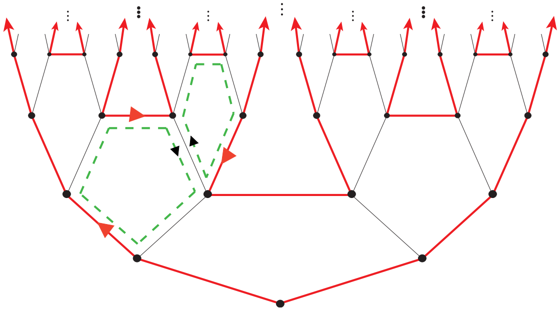

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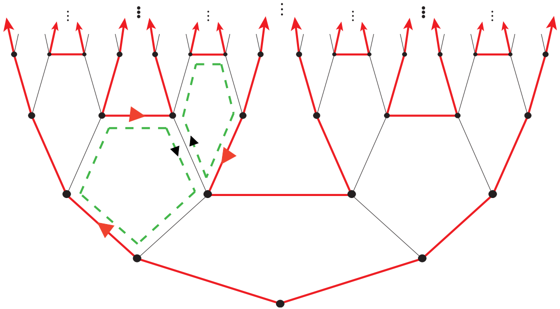

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OK if $\sum r(e) < \infty$

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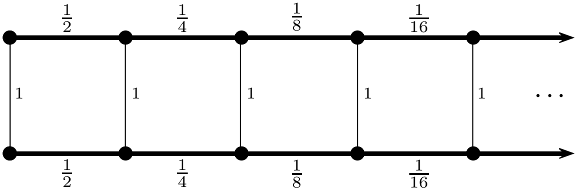

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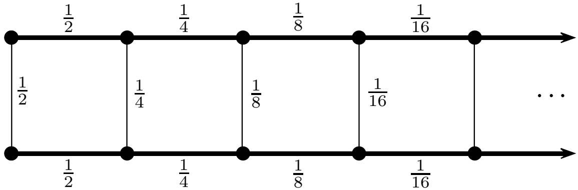

We need a tool that distinguishes edges with different resistances:

ℓ-*TOP*

- let $G = (V, E)$ be any graph
- give each edge a length $\ell(e)$
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Theorem (G '06 (easy))

If $\sum_{e \in E} \ell(e) < \infty$ *then* $|G|_\ell \approx |G|$.

The second tool

We need a tool that distinguishes edges with different resistances:

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- let $|G|_\ell$ be the completion of the corresponding metric space

Theorem (G '06 (easy))

If $\sum_{e \in E} r(e) < \infty$ *then* $|G|_r \approx |G|$.

Kirchhoff's cycle law for wild circles

Theorem (Diestel & G)

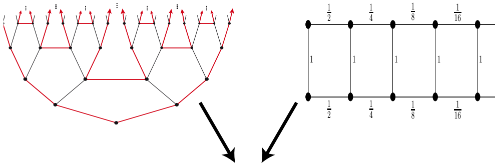
The circles of a electrical network N satisfy Kirchhoff's cycle law if the sum of the resistances in N is finite.

The Theorem

Theorem (G '08)

In a network with $\sum_{e \in E} r(e) < \infty$ there is a unique non-elusive flow with finite energy that satisfies Kirchhoff's cycle law.

The Theorem



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The Dirichlet Problem

Continuous version

Discrete version

The Dirichlet Problem

Continuous version

Let $X \subseteq \mathbb{R}^n$ be compact

Discrete version

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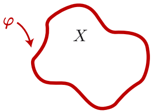
Discrete version

Let $X \subseteq \mathbb{R}^n$ be compact
Prescribe $\varphi : \partial X \to \mathbb{R}$



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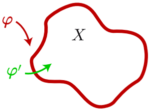

Extend to $\varphi' : X \to \mathbb{R}$ that is harmonic inside X

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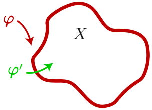

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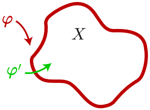

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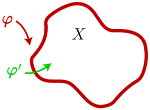

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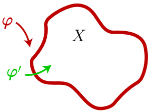

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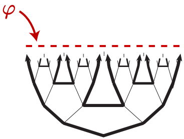

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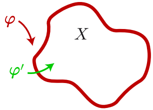

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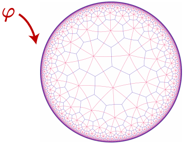

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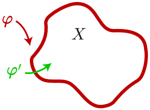

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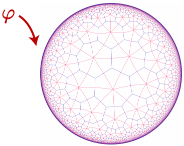

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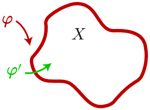

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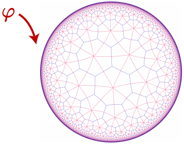

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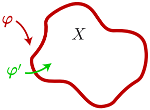

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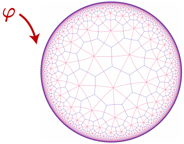

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i.e. satisfies Kirchhoff's node law

Studied intensively (Woess, Kaimanovich, Benjamini & Schramm)

The Dirichlet Problem

Problem

For evey assignment $r : E \to \mathbb{R}_+$ *(such that* $|G|_r$ *is compact) the Dirichlet problem is solvable for every continuous* $\phi : \partial|G|_r \to \mathbb{R}$*.*

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Interesting because:

Theorem

For every compact metric space X *there is a locally finite graph* G *and* $r : E \to \mathbb{R}_+$ *such that* $X = \partial|G|_r$.

The Dirichlet Problem

Problem

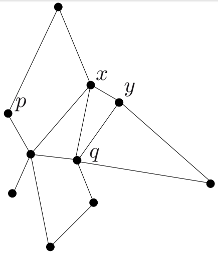
For evey assignment $r : E \to \mathbb{R}_+$ (such that $|G|_r$ is compact) the Dirichlet problem is solvable for every continuous $\phi : \partial|G|_r \to \mathbb{R}$.

The converse works:

Theorem

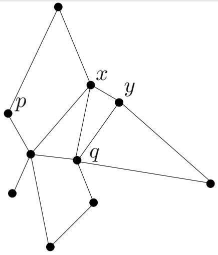
If $f : \vec{E} \to \mathbb{R}$ is a flow of finite energy in G satisfying Kirchhoff's cycle law then it is possible to extend the corresponding potentials continuously to $\partial|G|_r$.

Random Walks & Electrical networks



Random Walks & Electrical networks

Every edge e has a weight $c(e)$



Random Walks & Electrical networks

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Go from x to y with probability

$$P_{x \to y} := \frac{c(xy)}{c(x)}$$

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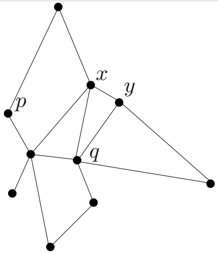

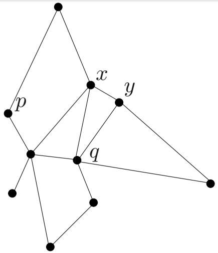

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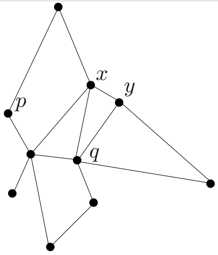

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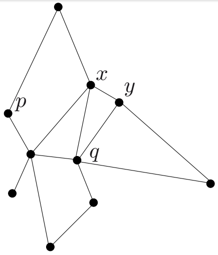

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Connect a source of voltage 1 to p, q

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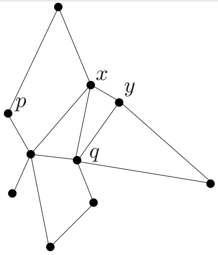

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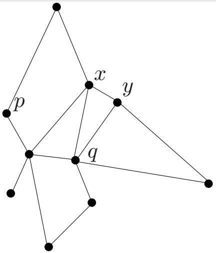

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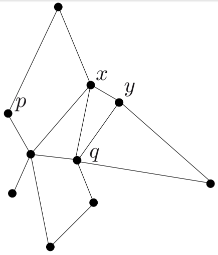

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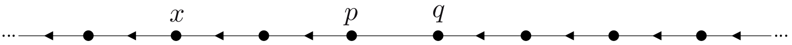

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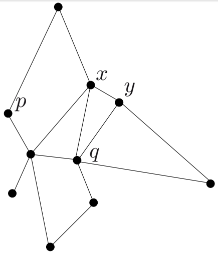

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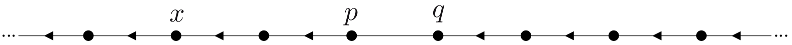
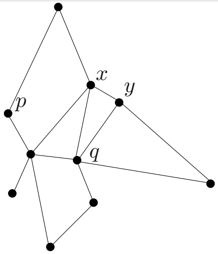


Random Walks & Electrical networks



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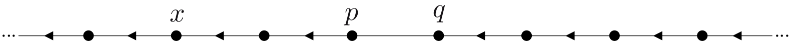
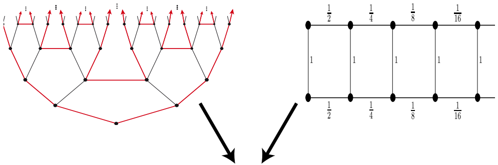


Problem

Define brownian motion on $|G|_\ell$

Summary



Theorem (G '08)

In a network with $\sum_{e\in E} r(e) < \infty$ there is a unique 'good' current

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Problem

Define brownian motion on $|G|_\ell$

Problem

Solve the Dirichlet Problem at the $|G|_\ell$ boundary