A new homology for infinite graphs and metric continua

Agelos Georgakopoulos

University of Warwick

Warwick, 14/3/13

Overview

- Wild spaces have a huge fundamental group π_1 and 1st homology group

Overview

- Wild spaces have a huge fundamental group π_1 and 1st homology group
  (1st Homology group H_1 = abelianization of π_1)

Overview

- Wild spaces have a huge fundamental group π_1 and 1st homology group (1st Homology group H_1 = abelianization of π_1)
- We are going to *tame* H_1 by removing some ‘redundancy’

Overview

- Wild spaces have a huge fundamental group π_1 and 1st homology group (1st Homology group H_1 = abelianization of π_1)
- We are going to *tame* H_1 by removing some 'redundancy'
- ... using experience from infinite graph theory

Example: MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

$\mathcal{C}(G)$: the cycle space of $G = H_1(G)$ (simlicial or singular homology)= $Abel(\pi_1)$
simple: no edge appears in more than two generators.

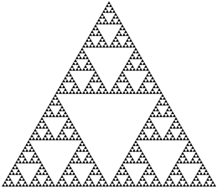
Example: MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

$\mathcal{C}(G)$: the cycle space of $G = H_1(G)$ (simlicial or singular homology)= $Abel(\pi_1)$

simple: no edge appears in more than two generators.



?

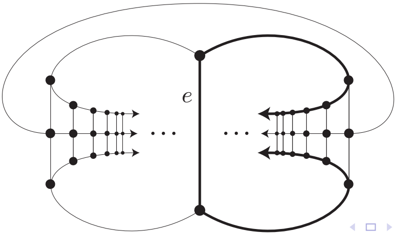
Example: MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

$\mathcal{C}(G)$: the cycle space of $G = H_1(G)$ (simlicial or singular homology)= $Abel(\pi_1)$

simple: no edge appears in more than two generators.



Example: MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

$\mathcal{C}(G)$: the cycle space of $G = H_1(G)$ (simlicial or singular homology)= $Abel(\pi_1)$
simple: no edge appears in more than two generators.

But using the right homology
(topological cycle space of Diestel & Kühn) ...:

Theorem (Bruhn & Stein '05)

... verbatim generalisation for locally finite G.

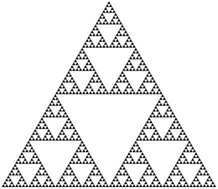
Example: MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

$\mathcal{C}(G)$: the cycle space of $G = H_1(G)$ (simlicial or singular homology)= $Abel(\pi_1)$

simple: no edge appears in more than two generators.



?

A new homology for metric spaces

Idea: put a natural distance function on $H_1(X)$ ...

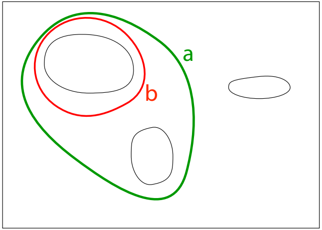
A new homology for metric spaces

Idea: put a natural distance function on $H_1(X)$ ...
... and identify elements at distance 0.

A new homology for metric spaces

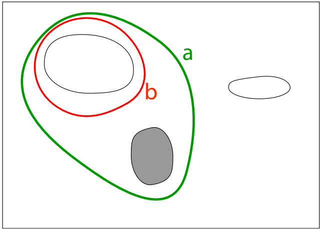
Idea: put a natural distance function on $H_1(X)$ ...

... and identify elements at distance 0.



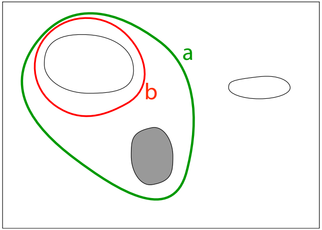
A new homology for metric spaces

Idea: put a natural distance function on $H_1(X)$ ...
... and identify elements at distance 0.



A new homology for metric spaces

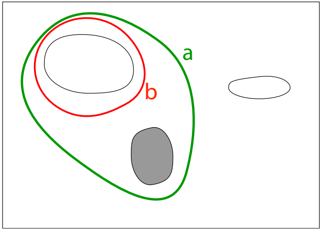
Idea: put a natural distance function on $H_1(X)$ ...
... and identify elements at distance 0.



$d(a,b) :=$ inf (area you need to make $a \approx b$)

A new homology for metric spaces

Idea: put a natural distance function on $H_1(X)$ ...
... and identify elements at distance 0.

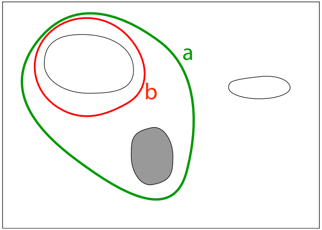


$d(a,b) :=$ inf (area you need to make $a \approx b$)

more precisely: $d(a,b) := \inf_{\substack{X \overset{\text{isom}}{\hookrightarrow} X' \\ a \approx b \text{ in } X'}} \text{area}(X' \backslash X)$

A new homology for metric spaces

Idea: put a natural distance function on $H_1(X)$ ...
... and identify elements at distance 0.



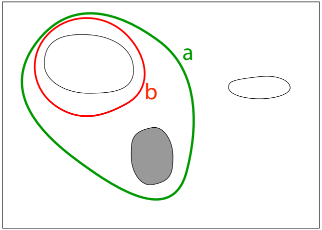
$d(a,b) :=$ inf (area you need to make $a \approx b$)

more precisely: $d(a,b) := \inf_{\substack{X \overset{\text{isom}}{\hookrightarrow} X' \\ a \approx b \text{ in } X'}} \text{area}(X' \setminus X)$

Let $H_1'(X) := H_1(X)/_{d=0}$

A new homology for metric spaces

Idea: put a natural distance function on $H_1(X)$ ...
... and identify elements at distance 0.

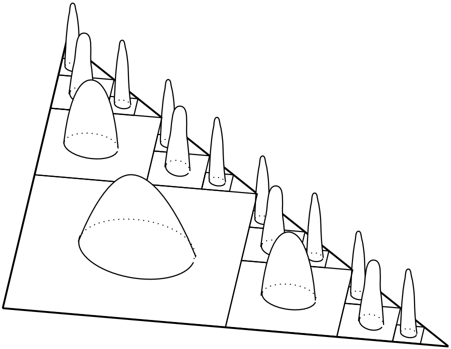


$d(a,b) :=$ inf (area you need to make $a \approx b$)

more precisely: $d(a,b) := \inf_{\substack{X \overset{\text{isom}}{\hookrightarrow} X' \\ a \approx b \text{ in } X'}} \text{area}(X' \backslash X)$

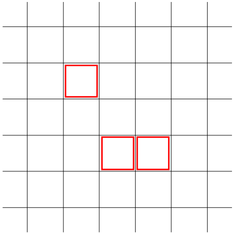
Let $H_1'(X) := H_1(X)/_{d=0}$
and, if you like, let $\widehat{H_1}(X)$ be its completeion.

Examples

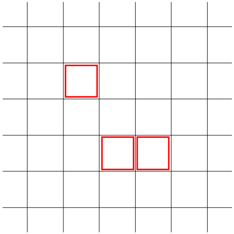


A wild space by Z. Virk & A. Zastrow.

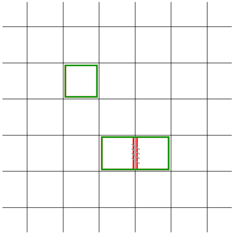
Cycle decompositions

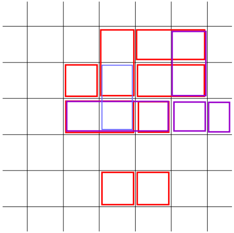


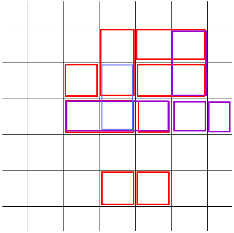
Cycle decompositions



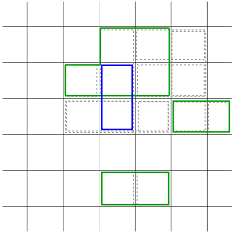
->

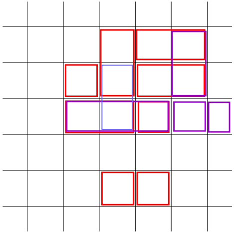




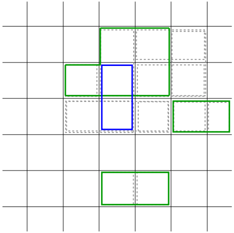


->





->

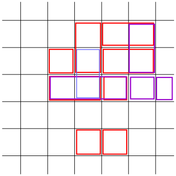


- Can you make a theorem out of this observation?

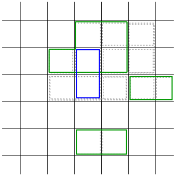
Cycle decompositions - finite graphs

Proposition

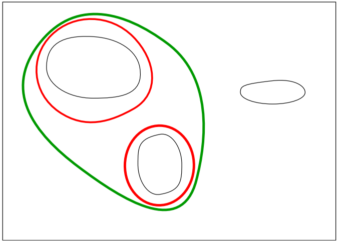
Every element of $\mathcal{C}(G)$ can be written as a union of a set of edge-disjoint cycles.



->



What about more continuous spaces?

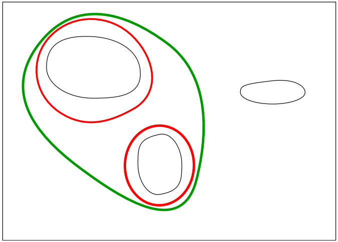


Proposition

Every element of $\mathcal{C}(G)$ can be written as a union of a set of edge-disjoint cycles.

?

What about more continuous spaces?



Proposition

Every element of $\mathcal{C}(G)$ can be written as a union of a set of edge-disjoint cycles.

?

Theorem (G' 09)

For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a σ-representative $(z_i)_{i \in \mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.

Cycle decompositions - infinite graphs

Theorem (Diestel & Kühn)

Every element of the topological cycle space $\mathcal{C}(G)$ of a locally finite graph G can be written as a union of a set of edge-disjoint circles.

Cycle decompositions - infinite graphs

Theorem (Diestel & Kühn)

Every element of the topological cycle space $\mathcal{C}(G)$ of a locally finite graph G can be written as a union of a set of edge-disjoint circles.

One of many classical theorems recently extended to infinite graphs using our new homology, the topological cycle space $\mathcal{C}(G)$ in an ongoing series of >30 papers by Diestel, Kühn, Bruhn, Stein, G, Sprüssel, Richter, Vella, et. al.

Proof sketch

Theorem (G' 09)

For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a σ-representative $(z_i)_{i\in\mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.

Proof sketch

Theorem (G' 09)

For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a σ-representative $(z_i)_{i \in \mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.

- Specify a subset of well-behaved elements of $\widehat{H_1}(X)$, called primitive elements;

Proof sketch

Theorem (G' 09)

For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a σ-representative $(z_i)_{i \in \mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.

- Specify a subset of well-behaved elements of $\widehat{H_1}(X)$, called primitive elements;
- Prove the statement for primitive elements;

Proof sketch

Theorem (G' 09)

For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a σ-representative $(z_i)_{i \in \mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.

- Specify a subset of well-behaved elements of $\widehat{H_1}(X)$, called primitive elements;
- Prove the statement for primitive elements;
- Show that every other element can be expressed as a sum of primitive elements.

Proof sketch

- Specify a subset of well-behaved elements of $\widehat{H_1}(X)$, called primitive elements;

Proof sketch

- Specify a subset of well-behaved elements of $\widehat{H_1}(X)$, called primitive elements;

We say that $C \in \widehat{H_1}(X)$ splits if there are $A, B \neq 0 \in \widehat{H_1}(X)$ with

$$C = A + B, \text{ and}$$
$$\ell(C) = \ell(A) + \ell(B).$$

Proof sketch

- Specify a subset of well-behaved elements of $\widehat{H_1}(X)$, called primitive elements;

We say that $C \in \widehat{H_1}(X)$ splits if there are $A, B \neq 0 \in \widehat{H_1}(X)$ with

$$C = A + B, \text{ and}$$
$$\ell(C) = \ell(A) + \ell(B).$$

...where $\ell(C)$ is the minimal length of 1-simplices needed to represent C.

Proof sketch

- Specify a subset of well-behaved elements of $\widehat{H_1}(X)$, called primitive elements;

We say that $C \in \widehat{H_1}(X)$ splits if there are $A, B \neq 0 \in \widehat{H_1}(X)$ with

$$C = A + B, \text{ and}$$
$$\ell(C) = \ell(A) + \ell(B).$$

...where $\ell(C)$ is the minimal length of 1-simplices needed to represent C.

Then C is primitive if it doesn't split.

Proof sketch

Theorem (G' 09)

For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a σ-representative $(z_i)_{i\in\mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.

- Specify a subset of well-behaved elements of $\widehat{H_1}(X)$, called primitive elements;
- Prove the statement for primitive elements;
- Show that every other element can be expressed as a sum of primitive elements.

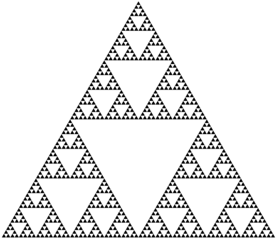
An intermediate result

Let $(\Gamma, +)$ be an abelian metrizable topological group, and suppose a function $\ell : \Gamma \to \mathbb{R}^+$ is given satisfying the following properties

- $\ell(a) = 0$ iff $a = 0$;
- $\ell(a + b) \leq \ell(a) + \ell(b)$ for every $a, b \in \Gamma$;
- if $b = \lim a_i$ then $\ell(b) \leq \liminf \ell(a_i)$;
- Some "isoperimetric inequality" holds: e.g. $d(a, 0) \leq U\ell^2(a)$ for some fixed U and for every $a \in \Gamma$.

Then every element of Γ is a (possibly infinite) sum of primitive elements.

The Conjecture

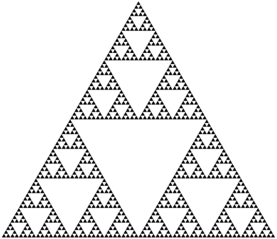


Theorem (MacLane '37)

A finite graph G is planar iff *$\mathcal{C}(G)$ has a simple generating set.*

?

The Conjecture



Theorem (MacLane '37)

A finite graph G is planar iff *$\mathcal{C}(G)$ has a simple generating set.*

?

Conjecture

Let X be a compact, 1–dimensional, locally connected, metrizable space that has no cut point. Then X is planar iff there is a simple set S of loops in X and a metric d inducing the topology of X so that the set $U := \{[\chi] \in \widehat{H_1}(X) \mid \chi \in S\}$ 'spans' $\widehat{H_1}(X)$.

ℓ-TOP

ℓ-TOP

- let $G = (V, E)$ be any graph

ℓ-TOP

- let $G = (V, E)$ be any graph
- give each edge a length $\ell(e)$

ℓ-TOP

- let $G = (V, E)$ be any graph
- give each edge a length $\ell(e)$
- this induces a metric: $d(v, w) := \inf\{\ell(P) \mid P \text{ is a } v\text{-}w \text{ path}\}$

ℓ-*TOP*

- let $G = (V, E)$ be any graph
- give each edge a length $\ell(e)$
- this induces a metric: $d(v, w) := \inf\{\ell(P) \mid P \text{ is a } v\text{-}w \text{ path}\}$
- let $|G|_\ell$ be the completion of the corresponding metric space

ℓ-TOP

- let $G = (V, E)$ be any graph
- give each edge a length $\ell(e)$
- this induces a metric: $d(v, w) := \inf\{\ell(P) \mid P \text{ is a } v\text{-}w \text{ path}\}$
- let $|G|_\ell$ be the completion of the corresponding metric space

Theorem (G '06)

*If $\sum_{e \in E(G)} \ell(e) < \infty$ then $|G|_\ell \approx |G|$. ...*and $\widehat{H_1}$ coincides with the topological cycle space and with $\check{H}_1(X)$.

ℓ-TOP

- let $G = (V, E)$ be any graph
- give each edge a length $\ell(e)$
- this induces a metric: $d(v, w) := \inf\{\ell(P) \mid P \text{ is a } v\text{-}w \text{ path}\}$
- let $|G|_\ell$ be the completion of the corresponding metric space

Theorem (G '06, '09)

If $\sum_{e \in E(G)} \ell(e) < \infty$ *then* $|G|_\ell \approx |G|$, and $\widehat{H_1}$ coincides with the topological cycle space and with $\check{H}_1(X)$.

ℓ-TOP

- let $G = (V, E)$ be any graph
- give each edge a length $\ell(e)$
- this induces a metric: $d(v, w) := \inf\{\ell(P) \mid P \text{ is a } v\text{-}w \text{ path}\}$
- let $|G|_\ell$ be the completion of the corresponding metric space

Problem

Does every compact metrizable space X admit a metric such that $\widehat{H_1}(X) = \check{H}_1(X)$?

ℓ-TOP

- let $G = (V, E)$ be any graph
- give each edge a length $\ell(e)$
- this induces a metric: $d(v, w) := \inf\{\ell(P) \mid P \text{ is a } v\text{-}w \text{ path}\}$
- let $|G|_\ell$ be the completion of the corresponding metric space

Theorem (Bourdon & Pajot, ...)

For every compact metric space X there is a locally finite graph G and $\ell : E \to R_+$ such that the boundary of $|G|_\ell$ is isometric to X.

Applications of $|G|_\ell$ (ℓ-*TOP*)

Applications of $|G|_\ell$

Applications of $|G|_\ell$ (ℓ-*TOP*)

- used by Floyd to study Kleinian groups *(Invent. math. '80)*

Applications of $|G|_\ell$ (ℓ-*TOP*)

- used by Floyd to study Kleinian groups *(Invent. math. '80)*
- used by Benjamini and Schramm for Random Walks/harmonic functions/sphere Packings *(Invent. math. '96, Preprint '09)*

Applications of $|G|_\ell$ (ℓ-*TOP*)

- used by Floyd to study Kleinian groups *(Invent. math. '80)*
- used by Benjamini and Schramm for Random Walks/harmonic functions/sphere Packings *(Invent. math. '96, Preprint '09)*
- application in the study of the Cycle Space of an infinite graph (G & Sprüssel *Electr. J. Comb*)

Applications of $|G|_\ell$

Applications of $|G|_\ell$ (ℓ-*TOP*)

- used by Floyd to study Kleinian groups *(Invent. math. '80)*
- used by Benjamini and Schramm for Random Walks/harmonic functions/sphere Packings *(Invent. math. '96, Preprint '09)*
- application in the study of the Cycle Space of an infinite graph (G & Sprüssel *Electr. J. Comb*)
- applied to Electrical Networks (G, *JLMS '10*)

Applications of $|G|_\ell$ (ℓ-*TOP*)

- used by Floyd to study Kleinian groups *(Invent. math. '80)*
- used by Benjamini and Schramm for Random Walks/harmonic functions/sphere Packings *(Invent. math. '96, Preprint '09)*
- application in the study of the Cycle Space of an infinite graph (G & Sprüssel *Electr. J. Comb*)
- applied to Electrical Networks (G, *JLMS '10*)
- Carlson studied the Dirichlet Problem at the boundary *(Analysis on graphs and its applications)*

Applications of $|G|_\ell$ (ℓ-*TOP*)

- used by Floyd to study Kleinian groups *(Invent. math. '80)*
- used by Benjamini and Schramm for Random Walks/harmonic functions/sphere Packings *(Invent. math. '96, Preprint '09)*
- application in the study of the Cycle Space of an infinite graph (G & Sprüssel *Electr. J. Comb*)
- applied to Electrical Networks (G, *JLMS '10*)
- Carlson studied the Dirichlet Problem at the boundary *(Analysis on graphs and its applications)*
- used by Colin de Verdiere et. al. to study Laplace and Schrödinger operators

Applications of $|G|_\ell$ (ℓ-*TOP*)

- used by Floyd to study Kleinian groups *(Invent. math. '80)*
- used by Benjamini and Schramm for Random Walks/harmonic functions/sphere Packings *(Invent. math. '96, Preprint '09)*
- application in the study of the Cycle Space of an infinite graph (G & Sprüssel *Electr. J. Comb*)
- applied to Electrical Networks (G, *JLMS '10*)
- Carlson studied the Dirichlet Problem at the boundary *(Analysis on graphs and its applications)*
- used by Colin de Verdiere et. al. to study Laplace and Schrödinger operators

Applications of $|G|_\ell$ (ℓ-*TOP*)

- used by Floyd to study Kleinian groups *(Invent. math. '80)*
- used by Benjamini and Schramm for Random Walks/harmonic functions/sphere Packings *(Invent. math. '96, Preprint '09)*
- application in the study of the Cycle Space of an infinite graph (G & Sprüssel *Electr. J. Comb*)
- applied to Electrical Networks (G, *JLMS '10*)
- Carlson studied the Dirichlet Problem at the boundary *(Analysis on graphs and its applications)*
- used by Colin de Verdiere et. al. to study Laplace and Schrödinger operators

All above authors "discovered" $|G|_\ell$ independently!

Further Problems

- Generalise to higher dimensions

Further Problems

- Generalise to higher dimensions
- Generalise other graph-theoretical theorems to continua/fractals

Further Problems

- Generalise to higher dimensions
- Generalise other graph-theoretical theorems to continua/fractals
- Can you modify $\widehat{H_1}$ to obtain a homology that is invariant under homotopy–equivalence?

Further Problems

- Generalise to higher dimensions
- Generalise other graph-theoretical theorems to continua/fractals
- Can you modify $\widehat{H_1}$ to obtain a homology that is invariant under homotopy–equivalence?
- Try to ‘tame’ π_1 by similar methods

Further Problems

- Generalise to higher dimensions
- Generalise other graph-theoretical theorems to continua/fractals
- Can you modify $\widehat{H_1}$ to obtain a homology that is invariant under homotopy–equivalence?
- Try to ‘tame’ π_1 by similar methods
- Compute $\widehat{H_1}$ for your favourite space

Further Problems

- Generalise to higher dimensions
- Generalise other graph-theoretical theorems to continua/fractals
- Can you modify $\widehat{H_1}$ to obtain a homology that is invariant under homotopy–equivalence?
- Try to 'tame' π_1 by similar methods
- Compute $\widehat{H_1}$ for your favourite space

Further Problems

- Generalise to higher dimensions
- Generalise other graph-theoretical theorems to continua/fractals
- Can you modify $\widehat{H_1}$ to obtain a homology that is invariant under homotopy–equivalence?
- Try to ‘tame’ π_1 by similar methods
- Compute $\widehat{H_1}$ for your favourite space

Sources:

AG: “Cycle decompositions: from graphs to continua”, Advances in Mathematics, 229(2):935–967, 2012.

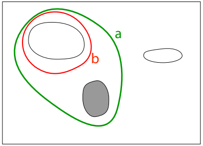
AG: “Graph topologies induced by edge lengths” Discrete Math., 311, 1523–1542, 2011.

These slides are available online

Summary

Theorem (G' 09)

For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a representative $(z_i)_{i\in\mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.



$$d(a,b) := \inf_{\substack{X \overset{\text{isom}}{\hookrightarrow} X' \\ a \approx b \text{ in } X'}} \text{area}(X' \backslash X). \text{ Let } H_1'(X) := H_1(X)/_{d=0}$$

Conjecture

Let X be a compact, 1–dimensional, locally connected, metrizable space that has no cut point. Then X is planar iff there is a simple set S of loops in X and a metric d inducing the topology of X so that the set $U := \{[\chi] \in \widehat{H_1}(X) \mid \chi \in S\}$ 'spans' $\widehat{H_1}(X)$.