From finite graphs to infinite; and beyond

Agelos Georgakopoulos

Mathematisches Seminar
Universität Hamburg

Göttingen, 5.3.2009

Things that go wrong in infinite graphs

Many finite theorems fail for infinite graphs:

Things that go wrong in infinite graphs

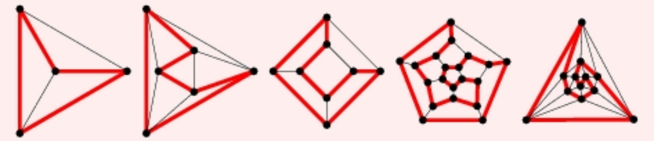
Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems

Hamilton cycles

Hamilton cycle: A cycle containing all vertices.

Some examples:



Things that go wrong in infinite graphs

Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems

Things that go wrong in infinite graphs

Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems
- Extremal graph theory

Things that go wrong in infinite graphs

Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems
- Extremal graph theory
- many others ...

Things that go wrong in infinite graphs

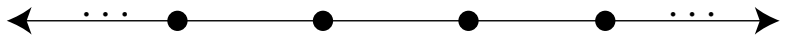
Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems
- Extremal graph theory
- many others ...

$\Rightarrow$ need more general notions

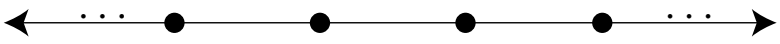
Spanning Double-Rays

Classical approach to 'save' Hamilton cycle theorems: accept double-rays as infinite cycles



Spanning Double-Rays

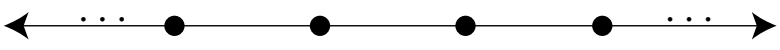
Classical approach to 'save' Hamilton cycle theorems: accept double-rays as infinite cycles



This approach only extends finite theorems in very restricted cases:

Spanning Double-Rays

Classical approach to 'save' Hamilton cycle theorems: accept double-rays as infinite cycles



This approach only extends finite theorems in very restricted cases:

Theorem (Tutte '56)

Every finite 4-connected planar graph has a Hamilton cycle

Spanning Double-Rays

Classical approach: accept double-rays as infinite cycles

This approach only extends finite theorems in very restricted cases:

Theorem (Yu '05)

Every locally finite 4-connected planar graph has a spanning double ray ...

Spanning Double-Rays

Classical approach: accept double-rays as infinite cycles

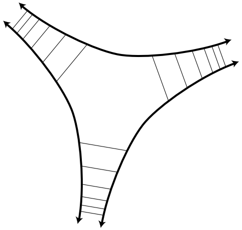
This approach only extends finite theorems in very restricted cases:

Theorem (Yu '05)

Every locally finite 4-connected planar graph has a spanning double ray ... unless *it is 3-divisible.*

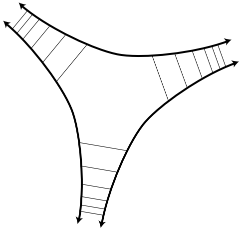
Compactifying by Points at Infinity

A 3-divisible graph



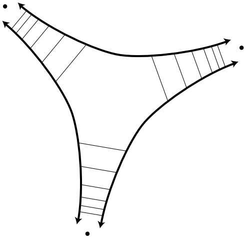
Compactifying by Points at Infinity

A 3-divisible graph
can have no spanning double ray



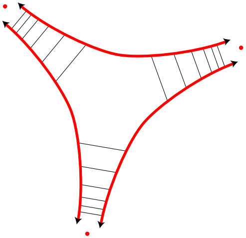
Compactifying by Points at Infinity

A 3-divisible graph
can have no spanning double ray



Compactifying by Points at Infinity

A 3-divisible graph
can have no spanning double ray



... but a Hamilton cycle?

Ends

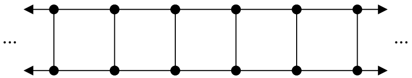
end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them

Ends

end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them

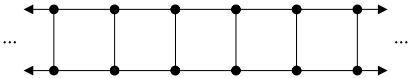


two ends

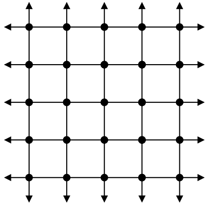
Ends

end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them



two ends

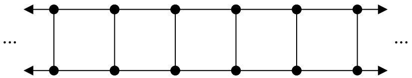


one end

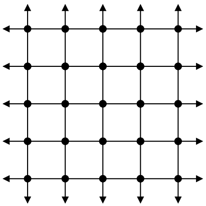
Ends

end: equivalence class of rays

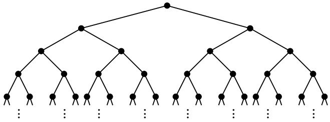
two rays are equivalent if no finite vertex set separates them



two ends

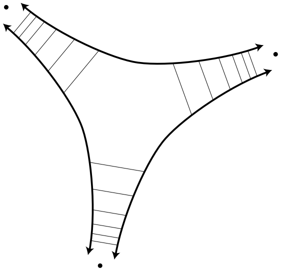


one end

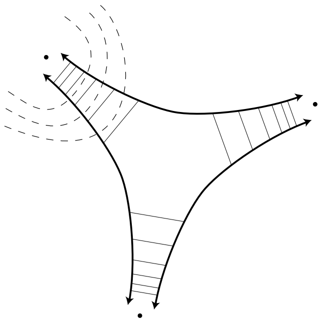


uncountably many ends

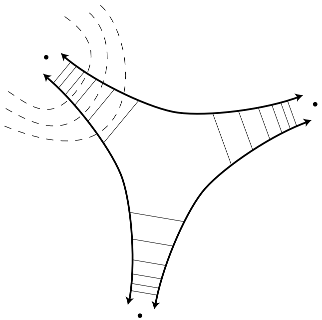
The End Compactification



The End Compactification



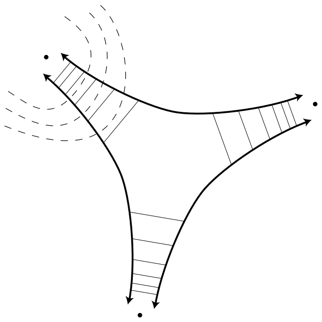
The End Compactification



Every ray converges to its end

The End Compactification

$$|G|$$



Every ray converges to its end

(Equivalent) definition of $|G|$

Give each edge e a length $\ell(e)$

(Equivalent) definition of $|G|$

Give each edge e a length $\ell(e)$

This naturally induces a metric d_ℓ on G

(Equivalent) definition of $|G|$

Give each edge e a length $\ell(e)$

This naturally induces a metric d_ℓ on G

Denote by $|G|_\ell$ the completion of (G, d_ℓ)

(Equivalent) definition of $|G|$

Give each edge e a length $\ell(e)$

This naturally induces a metric d_ℓ on G

Denote by $|G|_\ell$ the completion of (G, d_ℓ)

Theorem (G '06)

If $\sum_{e \in E(G)} \ell(e) < \infty$ then $|G|_\ell$ is homeomorphic to $|G|$.

Infinite Cycles

Circle:
A homeomorphic image of S^1 in $|G|$.

Infinite Cycles

Circle:
A homeomorphic image of S^1 in $|G|$.

Hamilton circle:
a circle containing all vertices

Infinite Cycles

Circle:
A homeomorphic image of S^1 in $|G|$.

Hamilton circle:
a circle containing all vertices (and all ends?)

Infinite Cycles

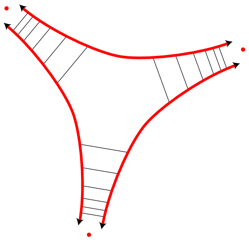
Circle:
A homeomorphic image of S^1 in $|G|$.

Hamilton circle:
a circle containing all vertices, and thus also all ends.

Infinite Cycles

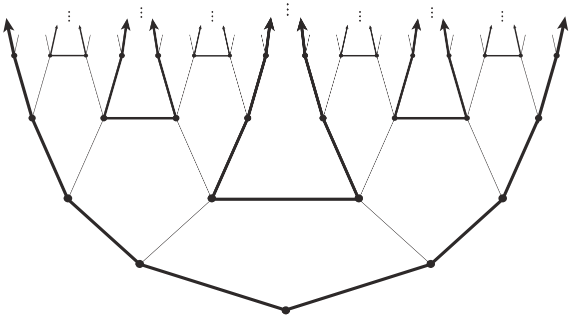
Circle:
A homeomorphic image of S^1 in $|G|$.

Hamilton circle:
a circle containing all vertices, and thus also all ends.



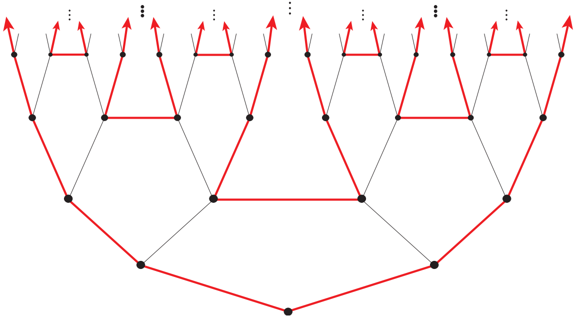
Infinite Cycles

Circle:
A homeomorphic image of S^1 in $|G|$.



Infinite Cycles

Circle:
A homeomorphic image of S^1 in $|G|$.



the wild circle of Diestel & Kühn

Fleischner's Theorem

Theorem (Fleischner '74)

The square of a finite 2-connected graph has a Hamilton cycle

Fleischner's Theorem

Theorem (Fleischner '74)

The square of a finite 2-connected graph has a Hamilton cycle

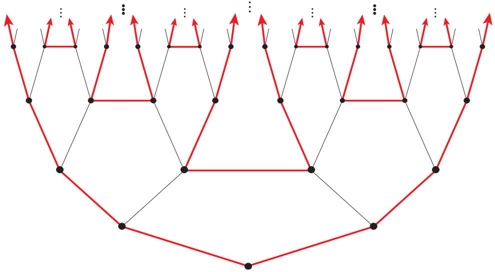
Theorem (Thomassen '78)

The square of a locally finite 2-connected 1-ended graph has a Hamilton circle.

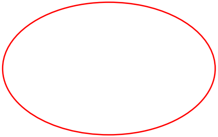
The Theorem

Theorem (G '06)

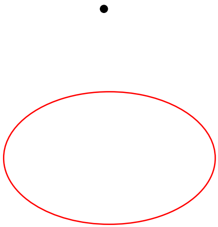
The square of any locally finite 2-connected graph has a Hamilton circle



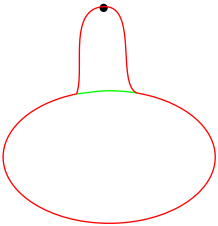
Proof?



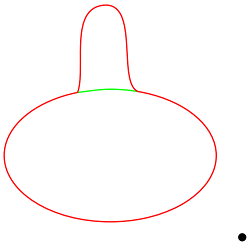
Proof?



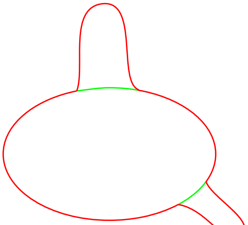
Proof?



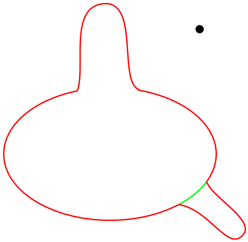
Proof?



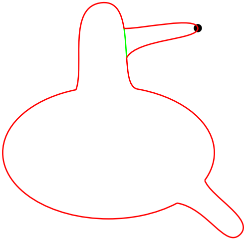
Proof?



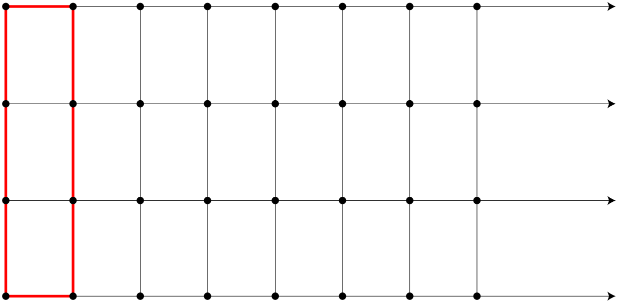
Proof?



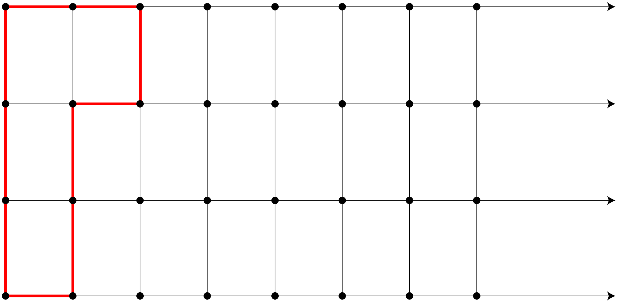
Proof?



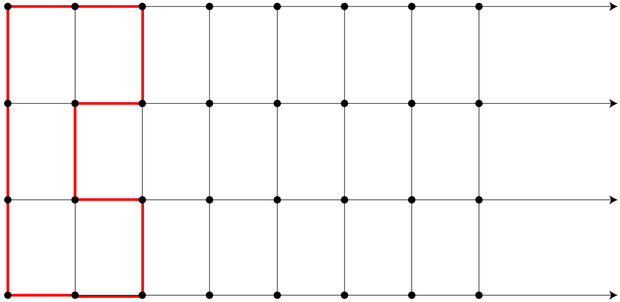
Proof?



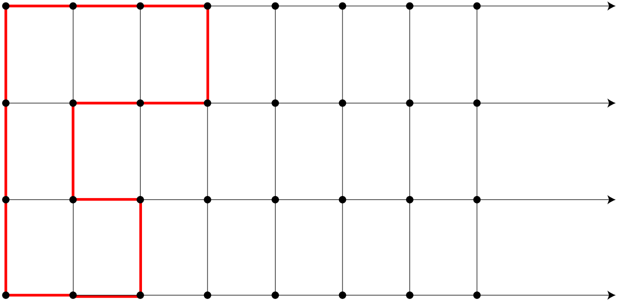
Proof?



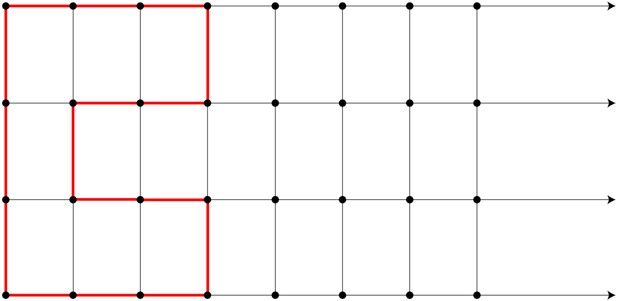
Proof?



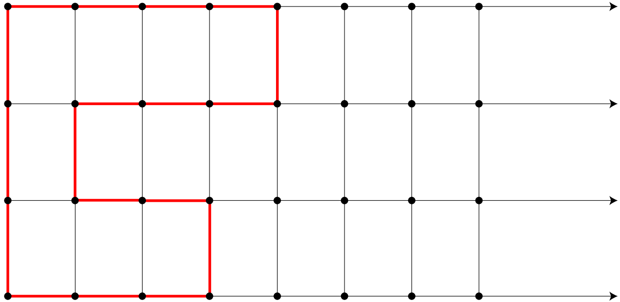
Proof?



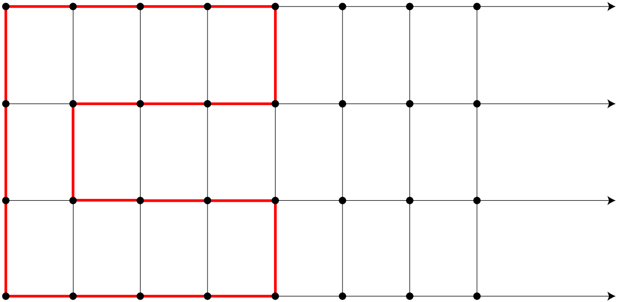
Proof?



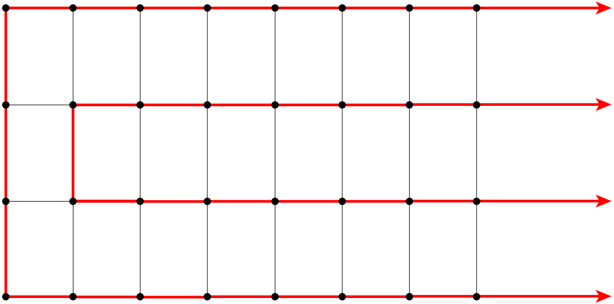
Proof?



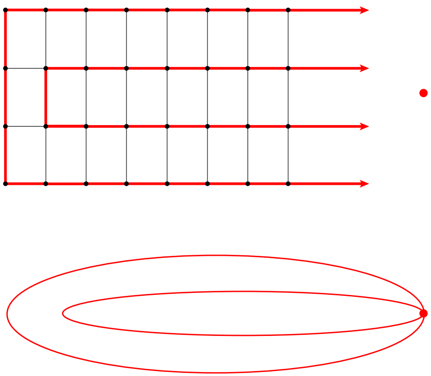
Proof?



Proof?

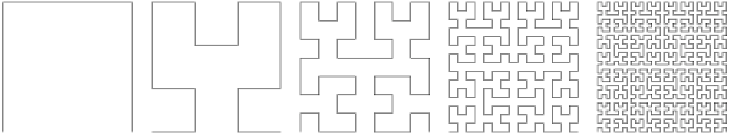


Proof?



Proof?

Hilbert's space filling curve:

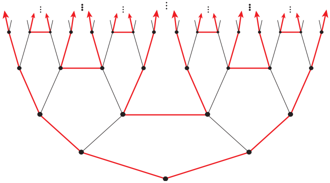


a sequence of injective curves with a non-injective limit

The Theorem

Theorem (G '06)

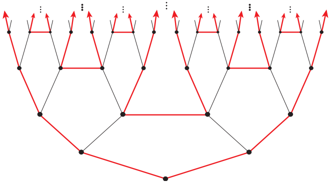
The square of any locally finite 2-connected graph has a Hamilton circle



The Theorem

Theorem (G '06)

The square of any locally finite 2-connected graph has a Hamilton circle



Corollary

Cayley graphs are "morally" hamiltonian.

Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser '59)

Does every finite connected Cayley graph have a Hamilton cycle?

Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser '59)

Does every finite connected Cayley graph have a Hamilton cycle?

Problem

Does every connected 1*-ended Cayley graph have a Hamilton circle?*

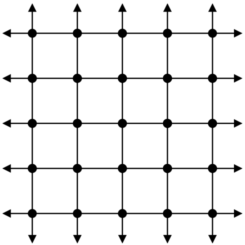
Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser '59)

Does every finite connected Cayley graph have a Hamilton cycle?

Problem

Does every connected 1*-ended Cayley graph have a Hamilton circle?*



Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser '59)

Does every finite connected Cayley graph have a Hamilton cycle?

Problem

Does every connected 1-ended Cayley graph have a Hamilton circle?

Problem

Prove that every connected Cayley graph of a finitely generated group Γ has a Hamilton circle unless Γ is the amalgamated product of more than k groups over a subgroup of order k.

Things that go wrong in infinite graphs

Many finite theorems fail for infinite graphs:

- Hamilton cycle theorems
- Extremal graph theory
- many others ...

Extremal Graph Theory

Theorem (Mader '72)

Any finite graph of minimum degree at least $4k$ has a k-connected subgraph.

k-connected means: you can delete any $k-1$ vertices and the graph will still be connected.

Extremal Graph Theory

Theorem (Mader '72)

Any finite graph of minimum degree at least $4k$ has a k-connected subgraph.

k-connected means: you can delete any $k-1$ vertices and the graph will still be connected.

Theorem (M. Stein '05)

Let $k \in \mathbb{N}$ and let G be a locally finite graph such that every vertex has degree at least $6k^2 - 5k + 3$ and every end has degree at least $6k^2 - 9k + 4$. Then G has a k-connected subgraph.

Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of cycles

Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of cycles

i.e., the first simplicial homology group of G.

Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of cycles

i.e., the first simplicial homology group of G.

The topological cycle space $\mathcal{C}(G)$ of a locally finite graph G is defined similarly but:

Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of cycles

i.e., the first simplicial homology group of G.

The topological cycle space $\mathcal{C}(G)$ of a locally finite graph G is defined similarly but:

- Allows edge sets of infinite circles;

Cycle Space

The cycle space $\mathcal{C}(G)$ of a finite graph:

- A vector space over $\mathbb{Z}_2$
- Consists of all sums of cycles

i.e., the first simplicial homology group of G.

The topological cycle space $\mathcal{C}(G)$ of a locally finite graph G is defined similarly but:

- Allows edge sets of infinite circles;
- Allows infinite sums (whenever well-defined).

The topological Cycle Space

Known facts:

- A connected graph has an Euler tour iff every edge-cut is even (Euler)
- G is planar iff $\mathcal{C}(G)$ has a simple generating set (MacLane)
- If G is 3-connected then its peripheral cycles generate $\mathcal{C}(G)$ (Tutte)

Generalisations:

Bruhn & Stein

Bruhn & Stein

Bruhn

MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

simple: no edge appears in more than two generators.

MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

simple: no edge appears in more than two generators.

Theorem (Bruhn & Stein'05)

... verbatim generalisation for locally finite G

$\mathcal{C}(G)$ and Singular Homology

There is a canonical homomorphism

$$f : H_1(|G|) \to \mathcal{C}(G)$$

$\mathcal{C}(G)$ and Singular Homology

There is a canonical homomorphism

$$f : H_1(|G|) \to \mathcal{C}(G)$$

Theorem (Diestel & Sprüssel '07)

f is surjective but not injective.

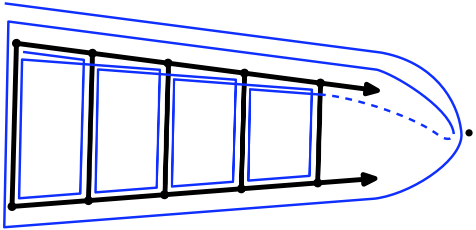
$\mathcal{C}(G)$ and Singular Homology

There is a canonical homomorphism

$$f : H_1(|G|) \to \mathcal{C}(G)$$

Theorem (Diestel & Sprüssel '07)

f is surjective but not injective.



$\mathcal{C}(G)$ and Singular Homology

Problem

Modify H_1 to obtain a homology theory that captures $\mathcal{C}(G)$ when applied to $|G|$ and generalises graph-theoretical theorems to arbitrary continua.

$\mathcal{C}(G)$ and Singular Homology

Problem

Modify H_1 to obtain a homology theory that captures $\mathcal{C}(G)$ when applied to $|G|$ and generalises graph-theoretical theorems to arbitrary continua.

In particular:

Problem

Characterise the continua embeddable in the plane

Electrical Networks

An electrical network is a graph G with an assignment of resistances $r : E(G) \to \mathbb{R}^+$, and two special vertices (source – sink) pumping a flow of constant value I into the network.

Electrical Networks

An electrical network is a graph G with an assignment of resistances $r : E(G) \to \mathbb{R}^+$, and two special vertices (source – sink) pumping a flow of constant value I into the network.

electrical flow: A flow satisfying Kirchhoff's second law (for finite cycles.

Electrical Networks

An electrical network is a graph G with an assignment of resistances $r : E(G) \to \mathbb{R}^+$, and two special vertices (source – sink) pumping a flow of constant value I into the network.

electrical flow: A flow satisfying Kirchhoff's second law (for finite cycles.

If G is finite then the electrical flow is *unique*, if it is infinite then there might be several; but:

Electrical Networks

An electrical network is a graph G with an assignment of resistances $r : E(G) \to \mathbb{R}^+$, and two special vertices (source – sink) pumping a flow of constant value I into the network.

electrical flow: A flow satisfying Kirchhoff's second law (for finite cycles.

If G is finite then the electrical flow is *unique*, if it is infinite then there might be several; but:

Theorem (G '08)

If $\sum_{e \in E} r(e) < \infty$ then there is a unique *non-elusive electrical flow of finite energy.*

energy := $\sum_{e \in E} i^2(e) r(e)$.