Discrete Riemann mapping and the Poisson boundary

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THE UNIVERSITY OF WARWICK

31/1/14

The Riemann mapping theorem

Theorem (Riemann? '1851, Carathéodory 1912)

For every simply connected open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$, *there is a bijective conformal map from* Ω *onto the open unit disk.*

The Riemann mapping theorem

Theorem (Riemann? '1851, Carathéodory 1912)

For every simply connected open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$, there is a bijective conformal map from Ω onto the open unit disk.

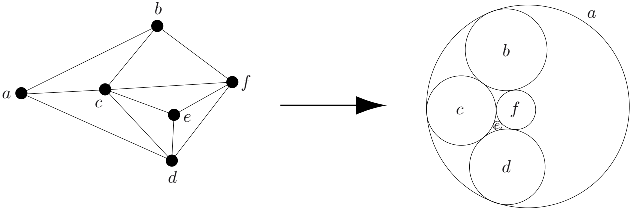
Theorem (Koebe 1908)

For every open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$ with finitely many boundary components, there is a bijective conformal map from Ω onto a circle domain.

The circle packing theorem

The Koebe-Andreev-Thurston circle packing theorem

For every finite planar graph G, there is a circle packing in the plane (or S^2) with nerve G.
The packing is unique (up to Möbius transformations) if G is a triangulation of S^2.



Circle Packing <=> Conformal map

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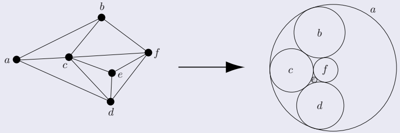

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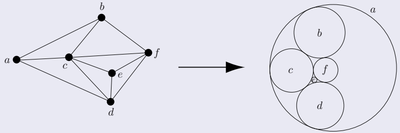

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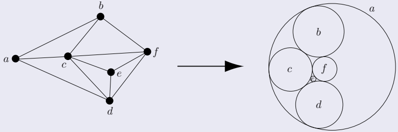

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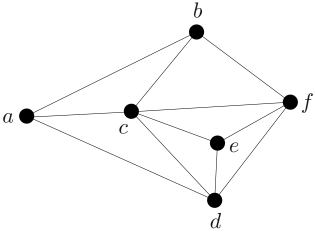

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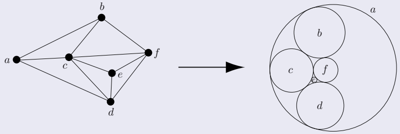

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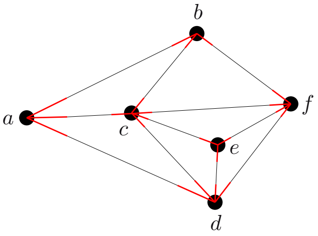

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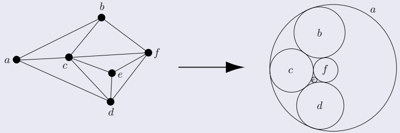

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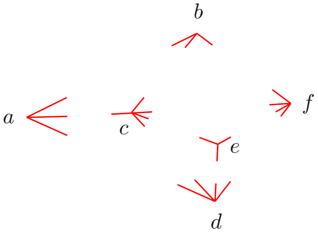

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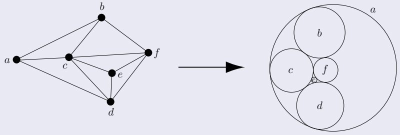

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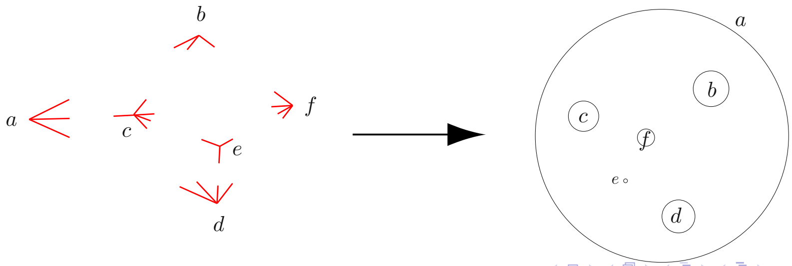

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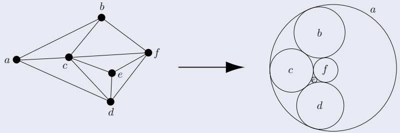

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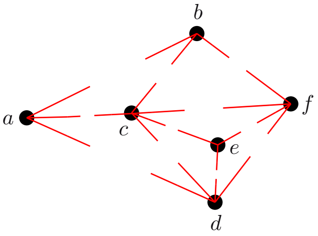

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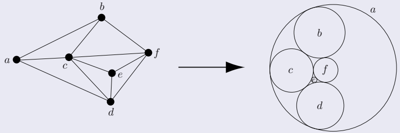

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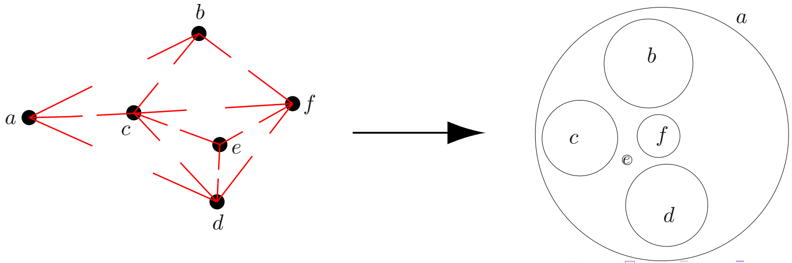

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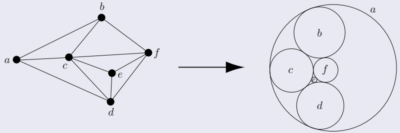

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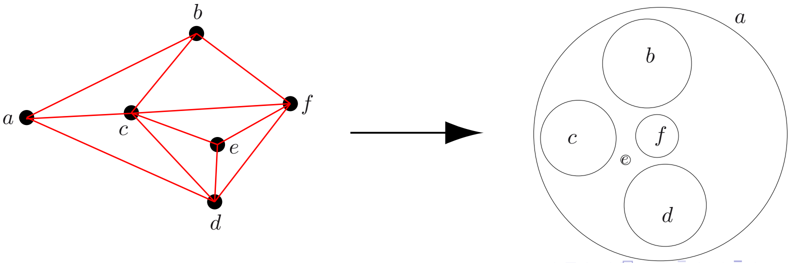

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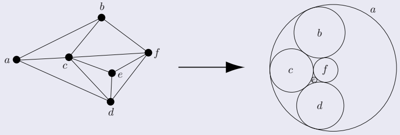

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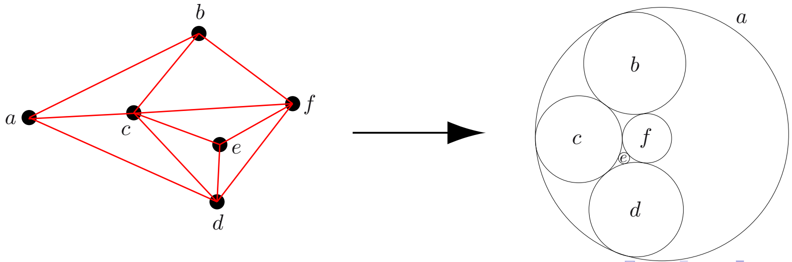

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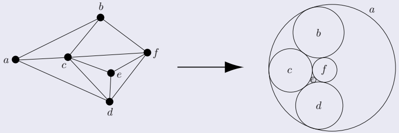

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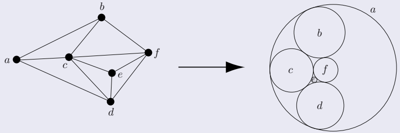

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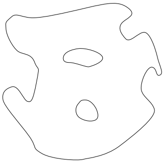

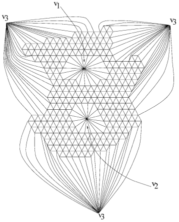

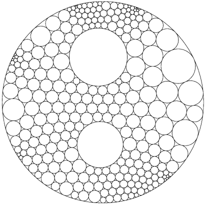

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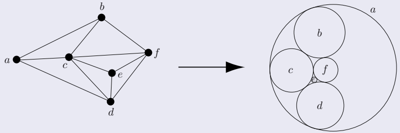

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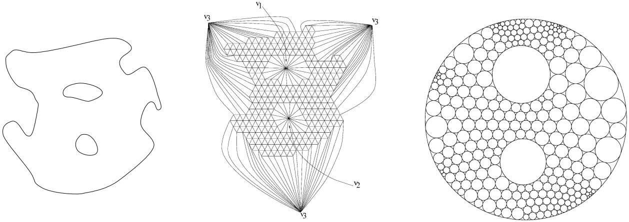
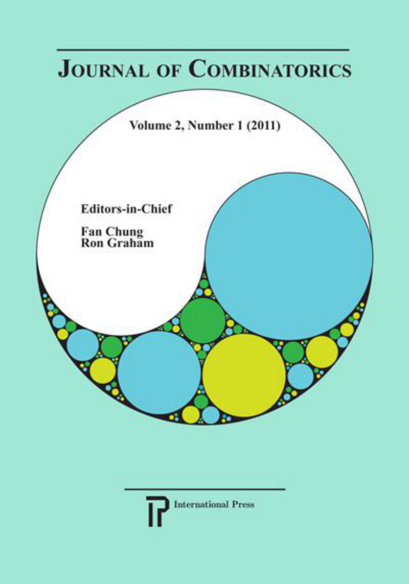


[*S. Rohde: “Oded Schramm: From Circle Packing to SLE”, ’10*]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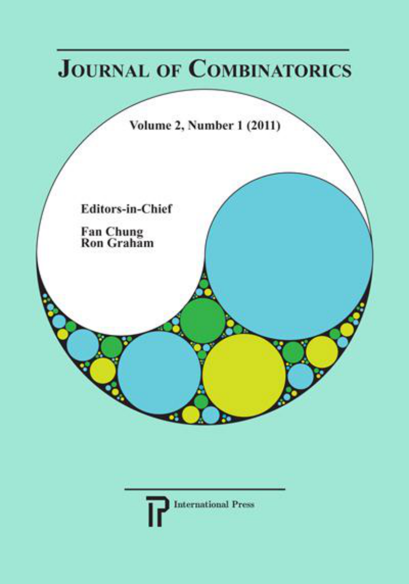
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Journal of Combinatorial Theory
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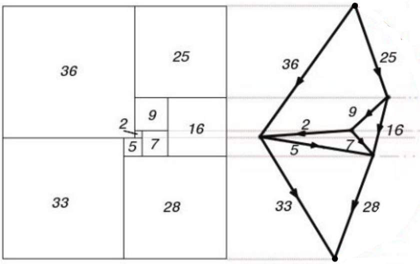
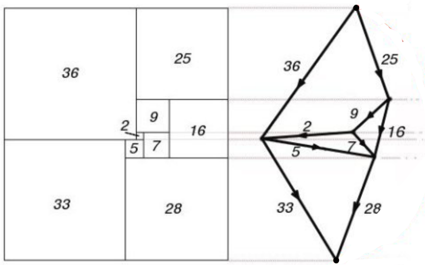
Square Tilings

Theorem (Brooks, Smith, Stone & Tutte '40)

... for every finite planar graph G, there is a square tiling with incidence graph G ...



Properties of square tilings



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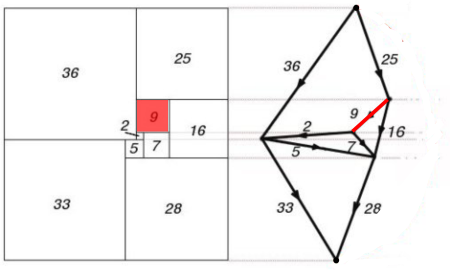

- every edge is mapped to a square;

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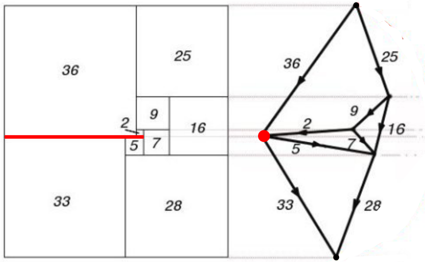

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- vertices correspond to horizontal segments tangent with their edges;

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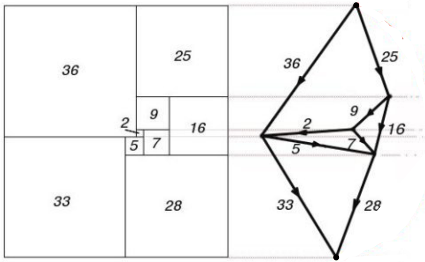

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- there is no overlap of squares, and no ‘empty’ space left;

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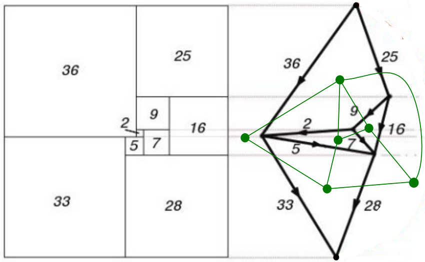

- every edge is mapped to a square;
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- the square tiling of the dual of G can be obtained from that of G by a 90° rotation.

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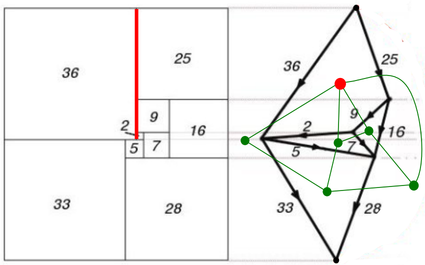

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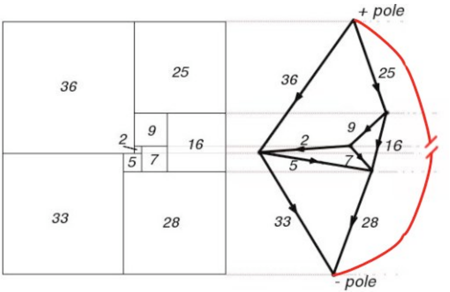
The construction of square tilings



- Think of the graph as an electrical network;

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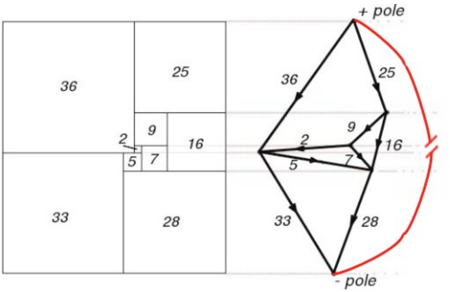

- Think of the graph as an electrical network;
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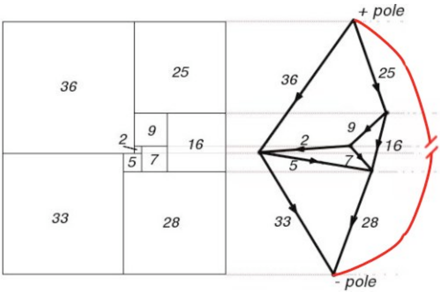

- Think of the graph as an electrical network;
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- let the square corresponding to edge e have side length the flow $i(e)$;

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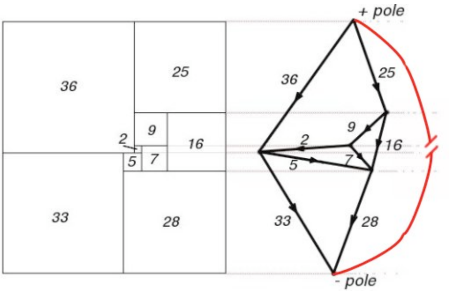

- Think of the graph as an electrical network;
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- let the square corresponding to edge e have side length the flow $i(e)$;
- place each vertex x at height equal to the potential $h(x)$;

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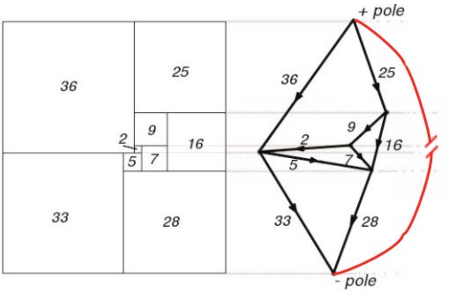

- Think of the graph as an electrical network;
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- let the square corresponding to edge e have side length the flow $i(e)$;
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- use a duality argument to determine the width coordinates.

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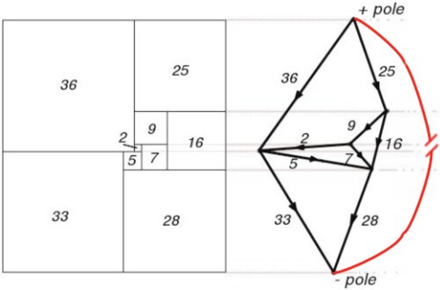

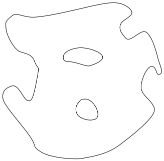

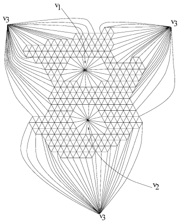

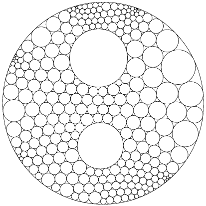

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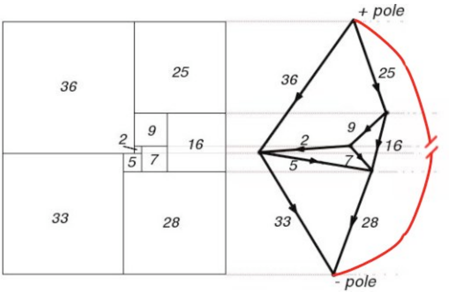

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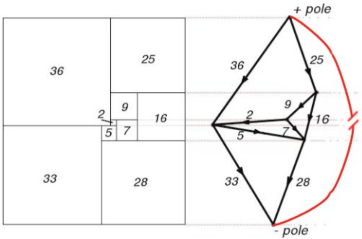

[*J. W. Cannon, W. J. Floyd, and W. R. Parry: "Squaring rectangles: The finite Riemann mapping theorem."*]

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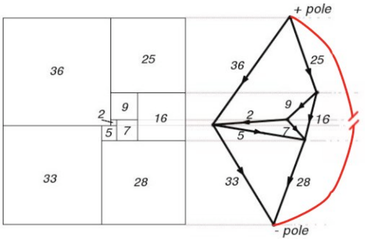

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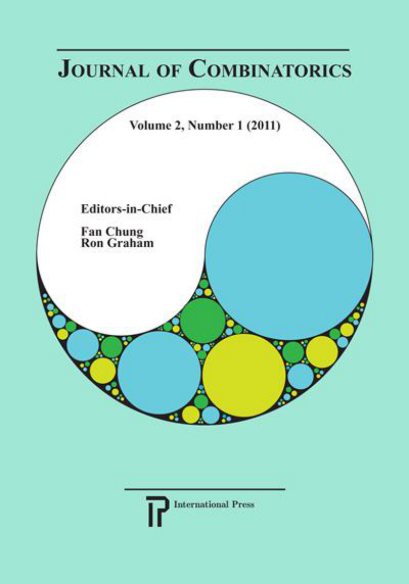
"... Riemann, like Klein in the passage quoted from Poincare, may have considered the quadrilateral as a metallic conducting plate with battery terminals connected to its 'top' and 'bottom'. "The current must pass" as Klein is supposed to have said. The current flow lines, connecting top to bottom, would have filled the quadrilateral from side to side one line through each point of the quadrilateral. Equipotential lines, connecting side to side, would likewise have filled the quadrilateral from top to bottom. The pair of families would meet one another orthogonally and give rectilinear flat coordinates for the quadrilateral."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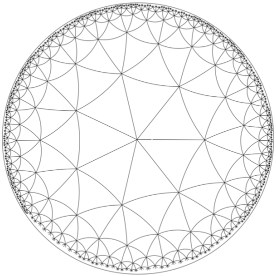
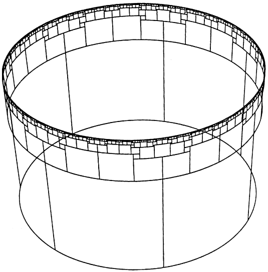
The square tilings of Benjamini & Schramm

Theorem (Benjamini & Schramm '96)

Every (transient) graph G of bounded degree that admits a uniquely absorbing embedding in the plane admits a square tiling.





$\mathcal{C}$

The square tilings of Benjamini & Schramm

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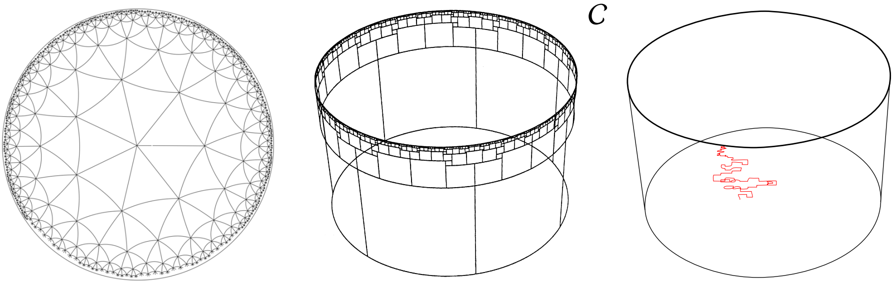
Every (transient) graph G of bounded degree that admits a uniquely absorbing embedding in the plane admits a square tiling. Moreover, random walk on G converges a. s. to a point in $\mathcal{C}$.



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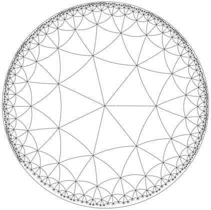
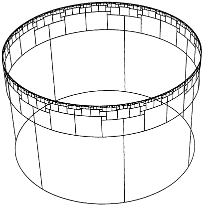
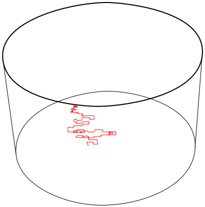
Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?

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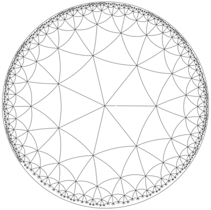
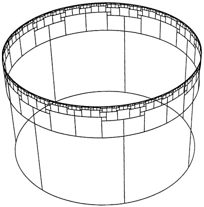
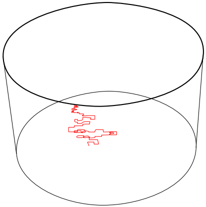






Theorem (G '12)

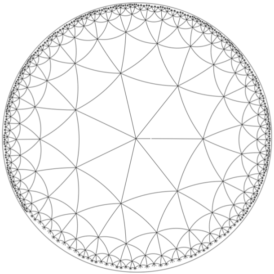
Yes.

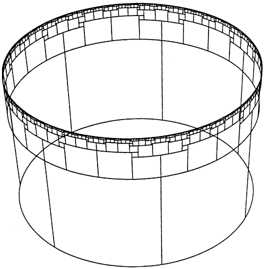
This is not about groups

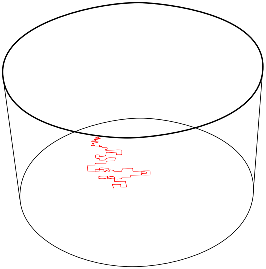
The theorem

Theorem (G '12)

For every bounded degree graph admitting a square tiling, the Poisson boundary coincides with C.







The theorem

Theorem (G '12)

For every bounded degree graph admitting a square tiling, the Poisson boundary coincides with C.

[Angel, Barlow, Gurel-Gurevich & Nachmias] recently identified the Poisson & Martin boundary of any bounded degree, transient, 1-ended triangulation of the plane with the boundary of its circle packing.

Sharp harmonic functions

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Thus they satisfy the σ-algebra axioms, except that there is no ground set.

The criterion

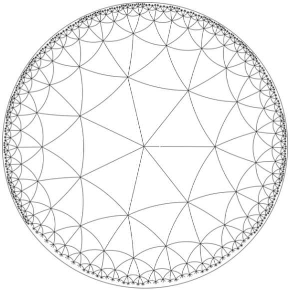
Theorem (G '12)

(Informal statement) Let M be a Markov chain. Any measurable space that can be used as the ground set of the 'σ-algebra' of sharp harmonic functions on M is a realisation of the Poisson boundary of M.

Corollaries

Conjecture (Northshield '93)

Let G be an accumulation-free plane, non-amenable graph with bounded vertex degrees. Then the Northshield circle of G is a realisation of its Poisson boundary.



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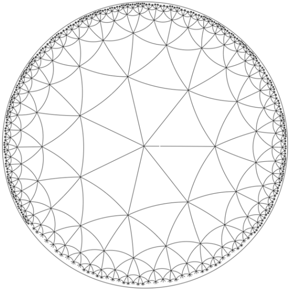


Theorem (G '13)

Indeed.

Corollaries

Corollary

Let G be an infinite, Gromov-hyperbolic, non-amenable, 1-ended, plane graph with bounded degrees and no infinite faces. Then the following five boundaries of G are canonically homeomorphic to each other:

- *the hyperbolic boundary*
- *the Martin boundary* [Ancona '88]
- *the boundary of the square tiling*
- *the Northshield circle, and*
- *the boundary* $\partial_{\cong}(G)$.

A conjecture

Conjecture (G)

Let M be a complete, simply connected Riemannian surface with Gaussian curvatures bounded between two negative constants. Let $f : M \to \mathbb{D}$ be a conformal map. Then for every 1-way infinite geodesic γ in M, the image $f(\gamma)$ converges to a point in the boundary $\mathbb{S}^1$ of $\mathbb{D}$, and this convergence determines a homeomorphism from the sphere at infinity of M to $\mathbb{S}^1$.

You can do more with
the Poisson boundary...

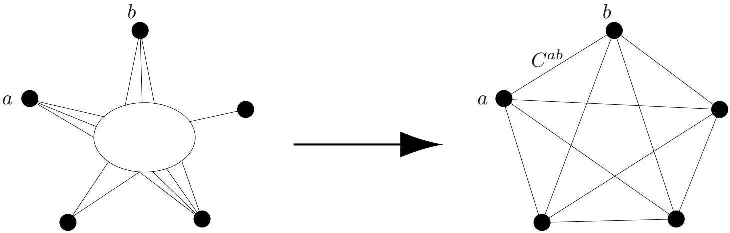
Energy and Douglas' formula

The classical Douglas formula

$$E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta$$

calculates the (Dirichlet) energy of a harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

Energy in finite electrical networks



$$E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 \, C^{ab},$$

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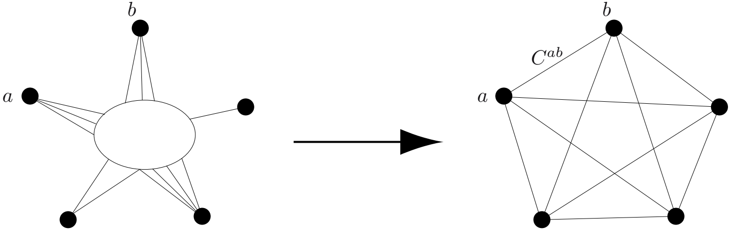

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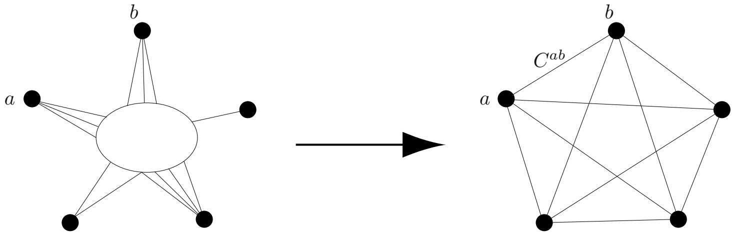
where $C^{ab} = d(a)\mathbb{P}_a(b)$



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Compare with Douglas: $E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta$

The energy of harmonic functions

Theorem (G & V. Kaimanovich '14+)

For every locally finite network G, there is a measure C on $\mathcal{P}^2(G)$ such that for every harmonic function u the energy $E(u)$ equals

$$\int_{\mathcal{P}^2} \left(\widehat{u}(\eta) - \widehat{u}(\zeta)\right)^2 dC(\eta, \zeta).$$

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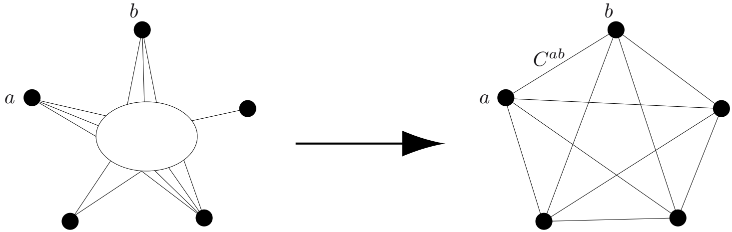
This is a discrete version of a result of [*Doob '62*] on Green spaces (or Riemannian manifolds), which generalises Douglas' formula $E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta$

Energy in finite electrical networks



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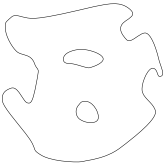
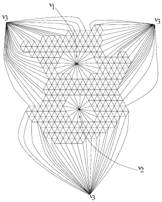
- Plans to generalise Sznitman's random interlacements ...

Summary

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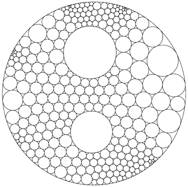

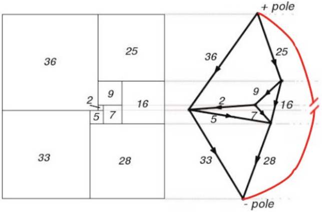
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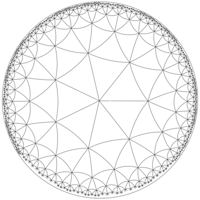

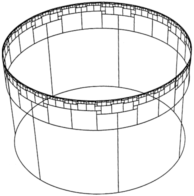

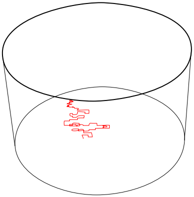

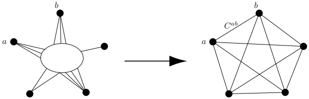
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