Discrete Riemann mapping and the Poisson boundary

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THE UNIVERSITY OF WARWICK

Lyon, 22/5/14

The Riemann mapping theorem

Theorem (Riemann? 1851, Carathéodory 1912)

For every simply connected open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$*, there is a bijective conformal map from* Ω *onto the open unit disk.*

The Riemann mapping theorem

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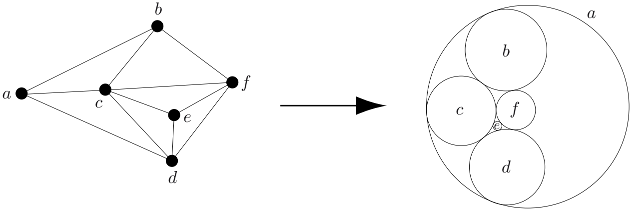
Theorem (Koebe 1920)

For every open set $\Omega \subsetneq \mathbb{C}, \Omega \neq \emptyset$ with finitely many boundary components, there is a bijective conformal map from Ω onto a circle domain.

The circle packing theorem

The Koebe-Andreev-Thurston circle packing theorem

For every finite planar graph G, there is a circle packing in the plane (or S^2) with nerve G.
The packing is unique (up to Möbius transformations) if G is a triangulation of S^2.



Circle Packing <=> Conformal map

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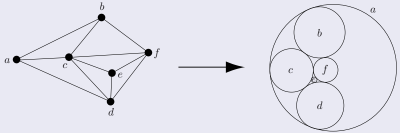

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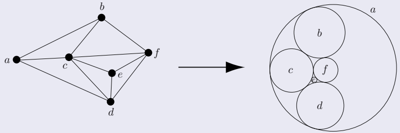

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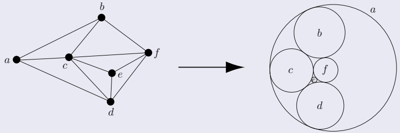

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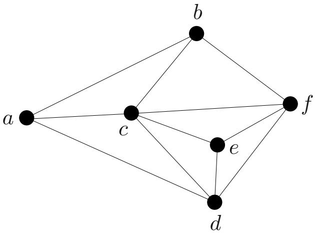

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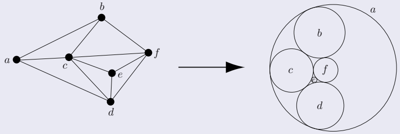

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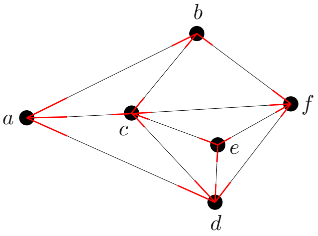

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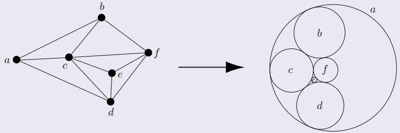

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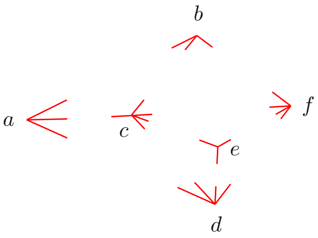

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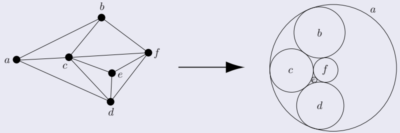

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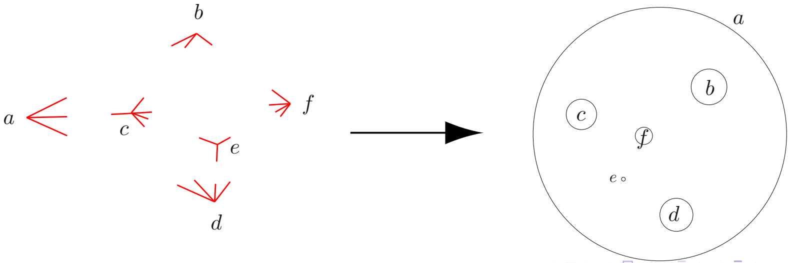

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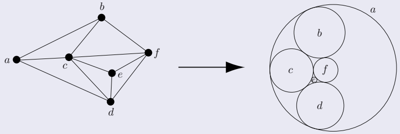

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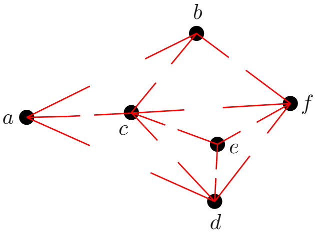

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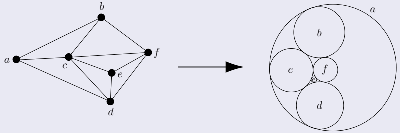

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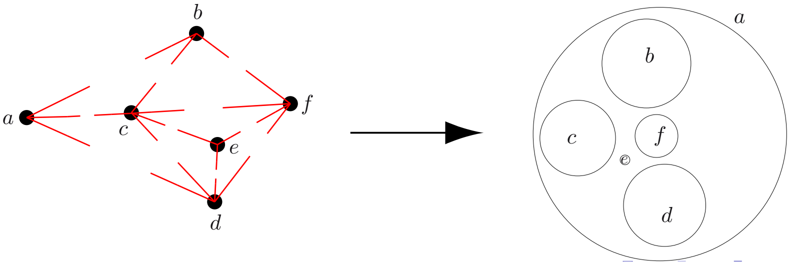

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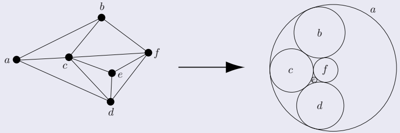

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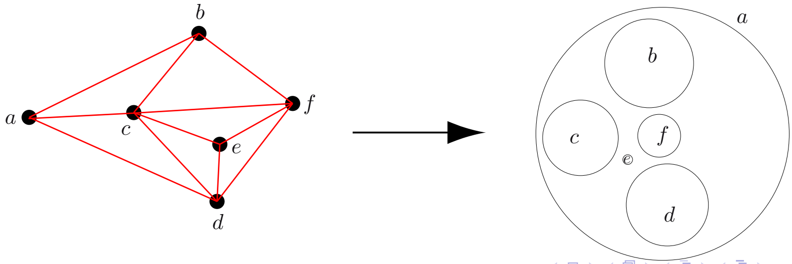

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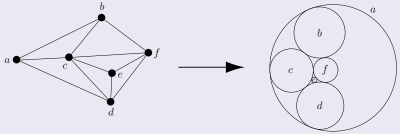

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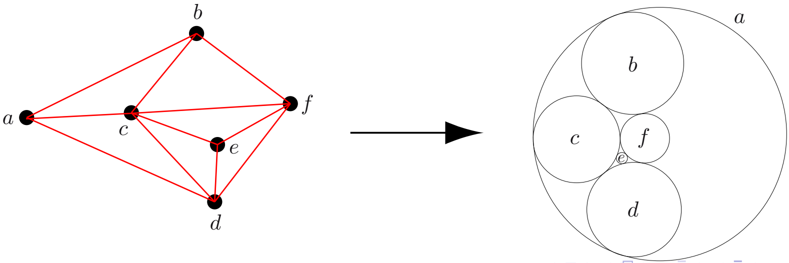

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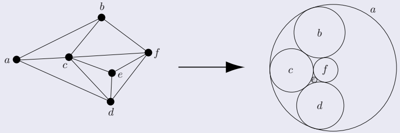

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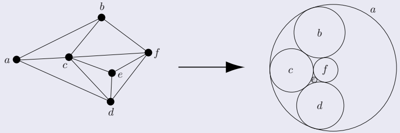

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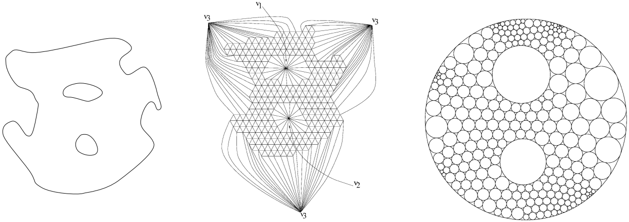

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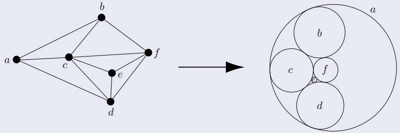

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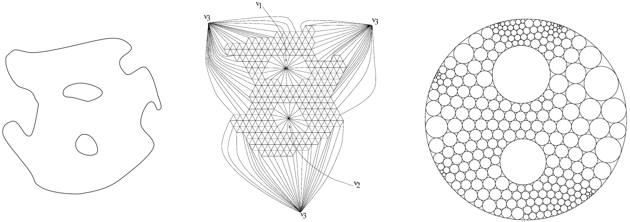
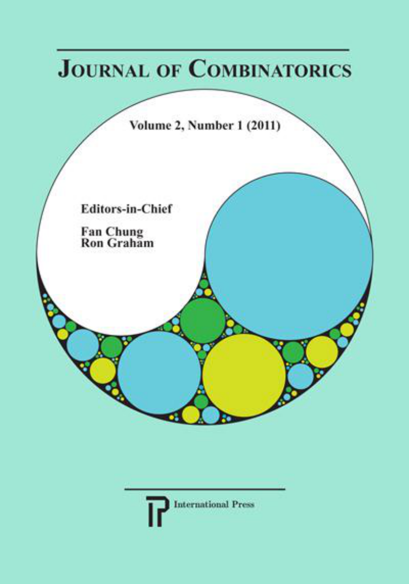


[*S. Rohde: "Oded Schramm: From Circle Packing to SLE", '10*]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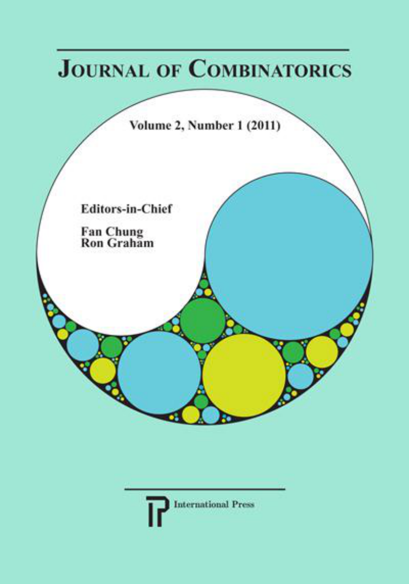
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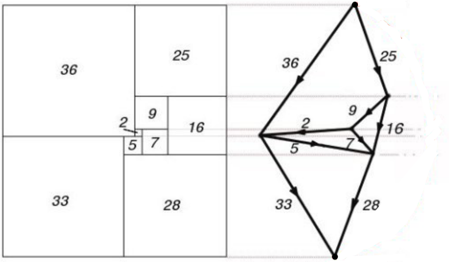
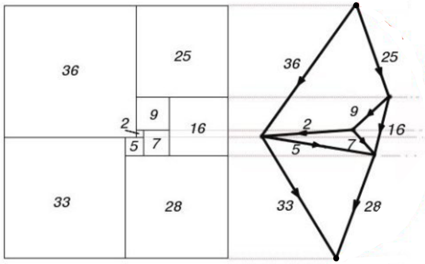
Square Tilings

Theorem (Brooks, Smith, Stone & Tutte '40)

... for every finite planar graph G, there is a square tiling with incidence graph G ...



Properties of square tilings



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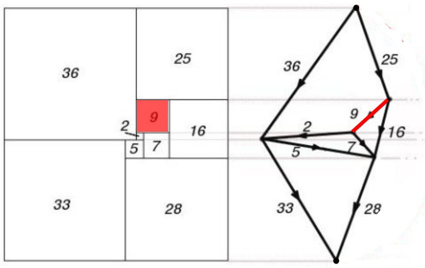

- every edge is mapped to a square;

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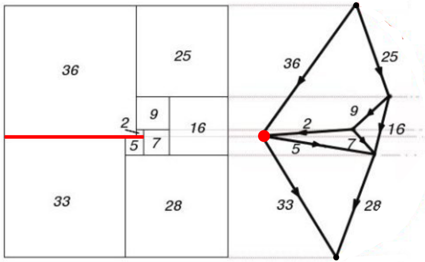

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- vertices correspond to horizontal segments tangent with their edges;

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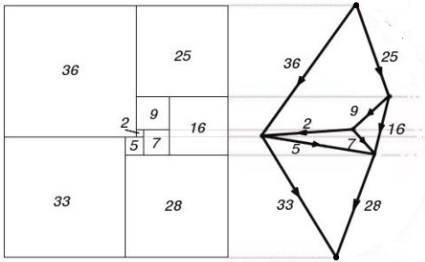

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- there is no overlap of squares, and no 'empty' space left;

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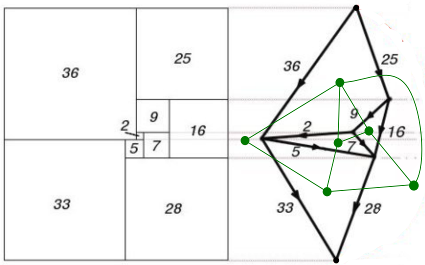

- every edge is mapped to a square;
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- the square tiling of the dual of G can be obtained from that of G by a 90° rotation.

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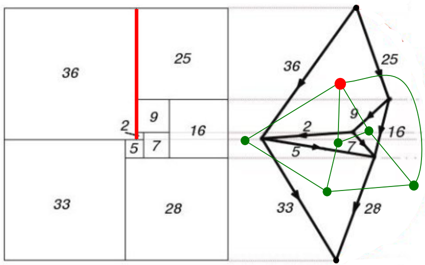

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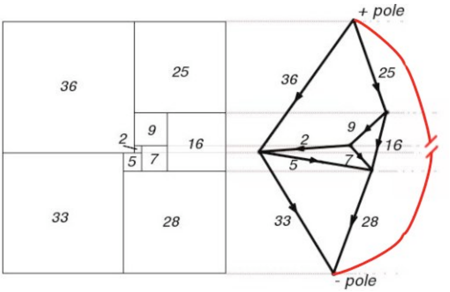

- Think of the graph as an electrical network;

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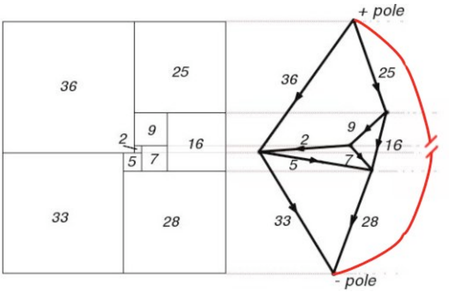

- Think of the graph as an electrical network;
- impose an electrical current from p to q;

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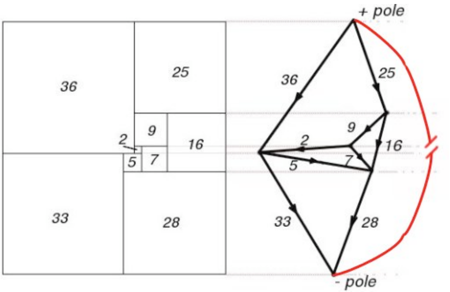

- Think of the graph as an electrical network;
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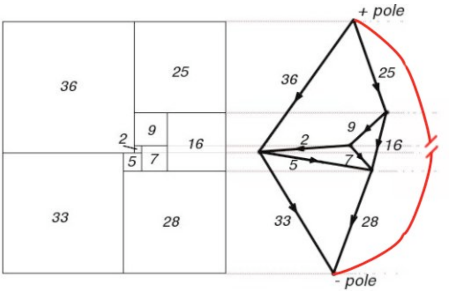

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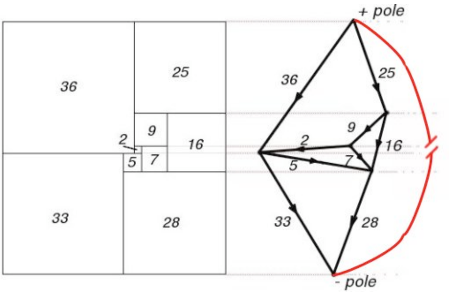

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- use a duality argument to determine the width coordinates.

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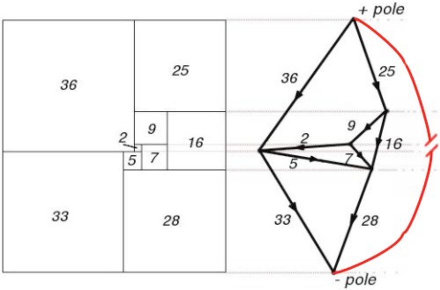

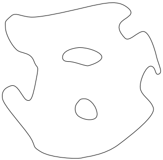

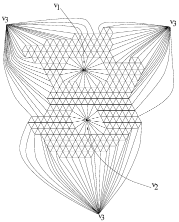

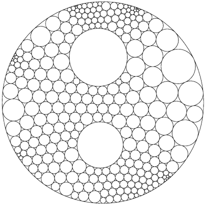

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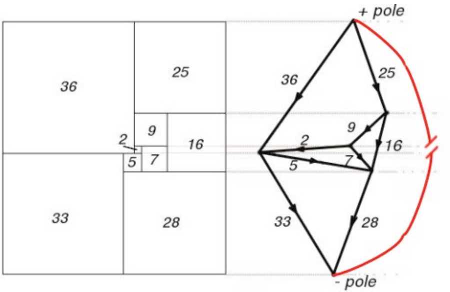

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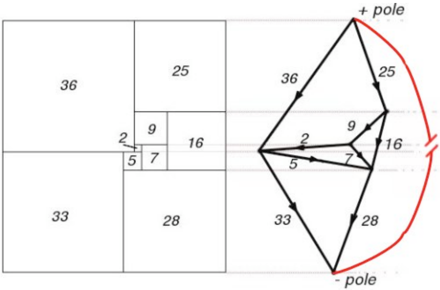
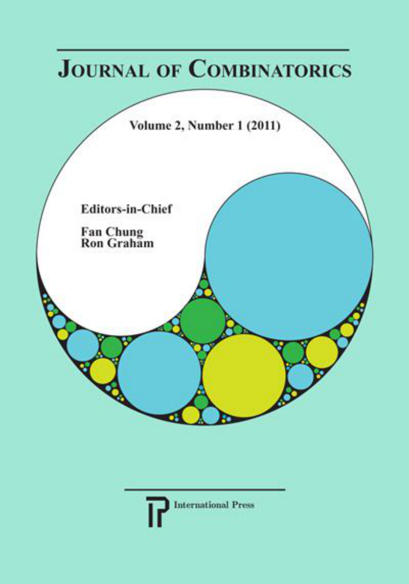


[*J. W. Cannon, W. J. Floyd, and W. R. Parry: "Squaring rectangles: The finite Riemann mapping theorem."*]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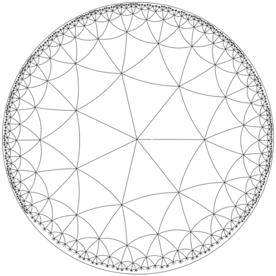
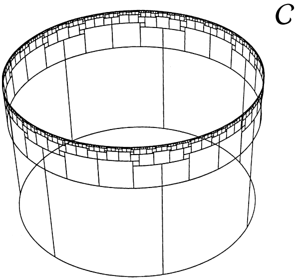
The square tilings of Benjamini & Schramm

Theorem (Benjamini & Schramm '96)

Every transient (infinite) graph G of bounded degree that has a uniquely absorbing embedding in the plane admits a square tiling.





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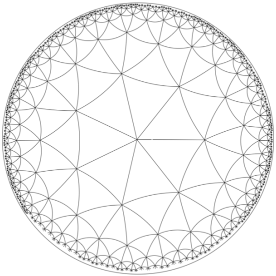
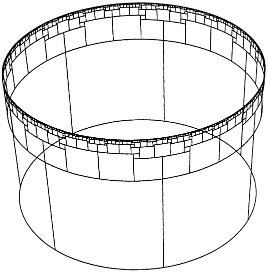
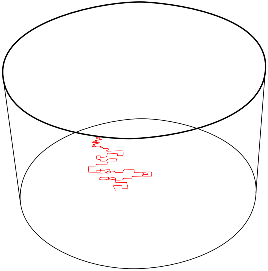




$\mathcal{C}$



The Poisson integral representation formula

The classical Poisson formula

$$h(z) = \int_0^{2\pi} \hat{h}(\theta) P(z,\theta) d\theta$$

where $P(z,\theta) := \frac{1-|z|^2}{|e^{i\theta}-z|^2}$,

recovers every continuous harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

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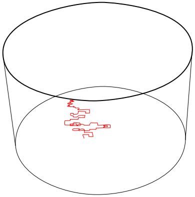
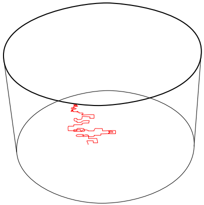


The boundary of the square tiling coincides with the Poisson boundary

Can the bounded harmonic functions on a plane graph G be expressed as a Poisson-like integral using C?



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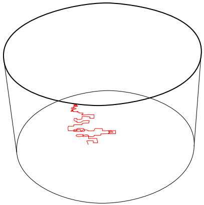

A function $h : V(G) \to \mathbb{R}$,
is **harmonic**, if $h(x) = \sum_{y \sim x} h(y)/d(x)$.

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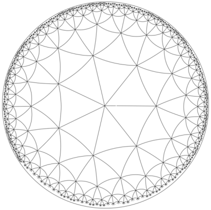
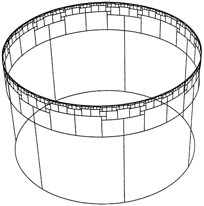
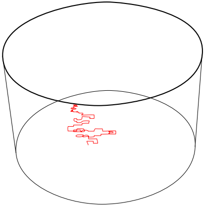
Question (Benjamini & Schramm '96)

Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?



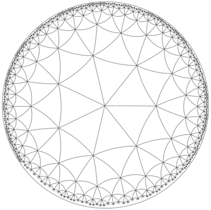


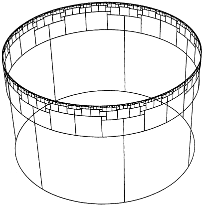


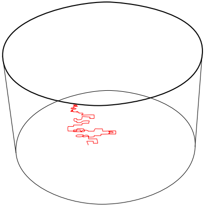
The boundary of the square tiling coincides with the Poisson boundary

Question (Benjamini & Schramm ’96)

Does the Poisson boundary of every graph as above coincide with the boundary of its square tiling?







Theorem (G ’12)

Yes!

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- this $\hat{h} \in L^\infty(\mathcal{P}_G)$ is unique up to modification on a null-set;
- conversely, for every $\hat{h} \in L^\infty(\mathcal{P}_G)$ the function $z \mapsto \int_{\mathcal{P}_G} \hat{h}(\eta) d\nu_z(\eta)$ is bounded and harmonic.

i.e. there is Poisson-like formula establishing an isometry between the Banach spaces $H^\infty(G)$ and $L^\infty(\mathcal{P}_G)$.

Bibliography

Selected work on the Poisson boundary

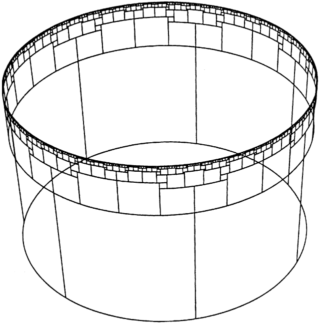
- Introduced by Furstenberg to study semi-simple Lie groups [*Annals of Math. '63*]
- Kaimanovich & Vershik give a general criterion using the entropy of random walk [*Annals of Probability '83*]
- Kaimanovich identifies the Poisson boundary of hyperbolic groups, and gives general criteria [*Annals of Math. '00*]
  General survey:
- Erschler: Poisson-Furstenberg Boundaries, Large-scale Geometry and Growth of Groups [*Proceedings of ICM 2010*]

Textbooks:
Woess: Random Walks on Infinite Graphs and Groups
Lyons & Peres: Probability on Trees and Networks

The theorem

Theorem (G '12)

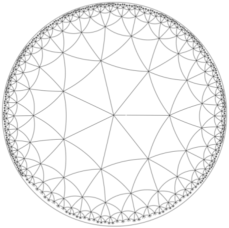
For every bounded degree graph admitting a square tiling, the Poisson boundary coincides with C.

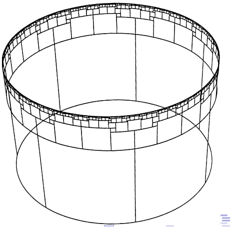


Probabilistic interpretation of the tiling

Lemma (G '12)

Let C be a 'horizontal' circle in the tiling T of G, and let B the set of points of G at which C 'dissects' T. Then the widths of the points of B in T coincide with the probability distribution of the first visit to B by brownian motion on G starting at o.





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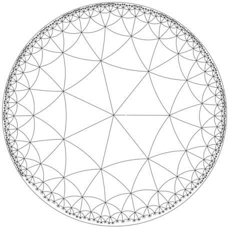
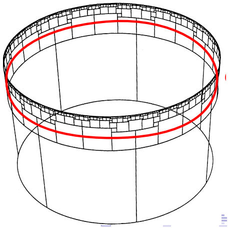
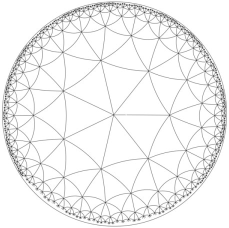
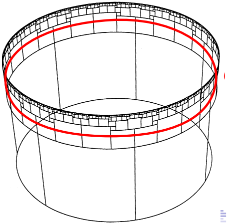




Lemma

For every 'meridian' M in T, the probability that brownian motion on G starting at o will 'cross' M clockwise equals the probability to cross M counter-clockwise.





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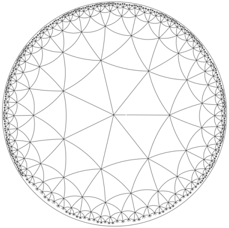
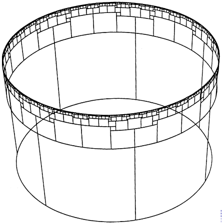
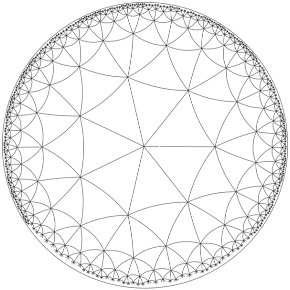




A corollary

Conjecture (Northshield '93)

Let G be an accumulation-free plane, non-amenable graph with bounded vertex degrees. Then the Northshield circle of G is a realisation of its Poisson boundary.



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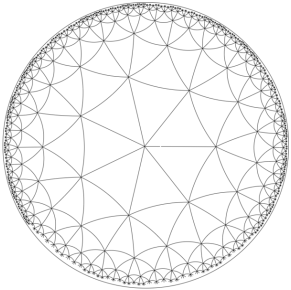

Theorem (G '13)

Indeed.

Hyperbolic planar graphs

Theorem (G '13)

Let G be an infinite, Gromov-hyperbolic, non-amenable, 1-ended, plane graph with bounded degrees and no infinite faces. Then the following 5 boundaries of G (and the corresponding compactifications of G) are canonically homeomorphic to each other:

- *the hyperbolic boundary*
- *the Martin boundary [Ancona]*
- *the boundary of the square tiling*
- *the Northshield circle $\partial_{\sim}(G)$ and*
- *the transience boundary $\partial_{\simeq}(G)$ [Northshield].*

Open problems

Conjecture (G)

Let M be a complete, simply connected Riemannian surface with sectional curvatures bounded between two negative constants. Let $f : M \to \mathbb{D}$ be a conformal map. Then for every 1-way infinite geodesic γ in M, the image $f(\gamma)$ converges to a point in the boundary $\mathbb{S}^1$ of $\mathbb{D}$, and this convergence determines a homeomorphism from the sphere at infinity of M to $\mathbb{S}^1$.

Open problems

Problem

Is every planar graph with the Liouville property amenable?

–For Cayley graphs this is true even without planarity [Kaimanovich & Vershik];
–for general graphs it is false even assuming bounded degrees [e.g. Benjamini & Kozma].

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Problem

Is there a planar, Gromov-hyperbolic graph with bounded degrees, no infinite faces, and the Liouville property?

Here come some
‘geometric’ random graphs

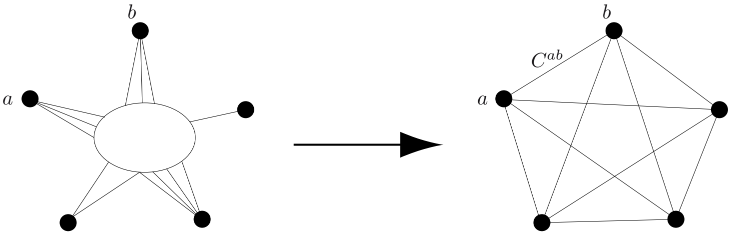
Energy and Douglas' formula

The classical Douglas formula

$$E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta$$

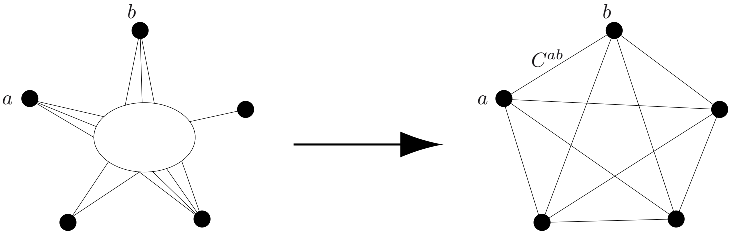
calculates the (Dirichlet) energy of a harmonic function h on $\mathbb{D}$ from its boundary values $\hat{h}$ on the circle $\partial\mathbb{D}$.

Energy in finite electrical networks



$$E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 \, C^{ab},$$

Energy in finite electrical networks



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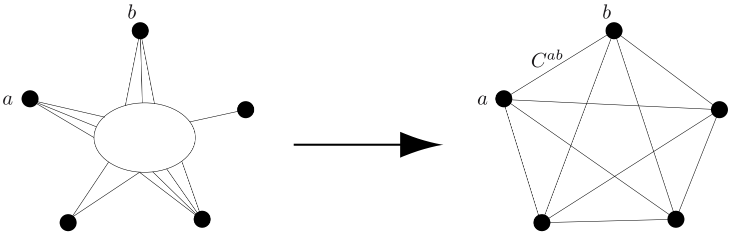

$$E(h) = \sum_{a,b \in B} (h(a) - h(b))^2 \, C^{ab},$$

$$\text{where } C^{ab} = d(a)\mathbb{P}_a(b)$$

Compare with Douglas: $E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta$

The energy of harmonic functions

Theorem (G & V. Kaimanovich '14+)

For every locally finite network G, there is a measure C on $\mathcal{P}^2(G)$ such that for every harmonic function u the energy $E(u)$ equals

$$\int_{\mathcal{P}^2} \left(\widehat{u}(\eta) - \widehat{u}(\zeta)\right)^2 dC(\eta, \zeta).$$

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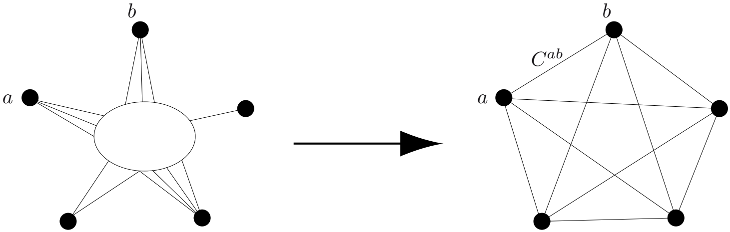
This is a discrete version of a result of [*Doob '62*] on Green spaces (or Riemannian manifolds), which generalises Douglas' formula $E(h) = \int_0^{2\pi} \int_0^{2\pi} (\hat{h}(\eta) - \hat{h}(\zeta))^2 \Theta(z, \eta) d\eta$

Energy in finite electrical networks



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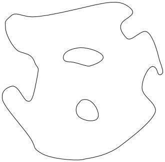
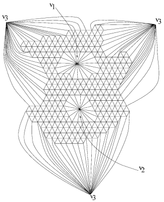
Summary





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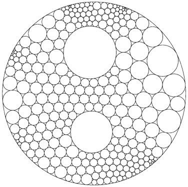

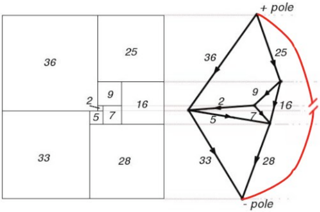

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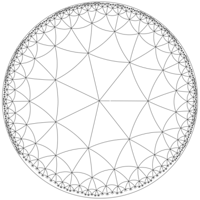

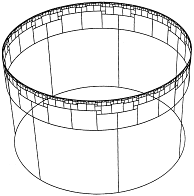

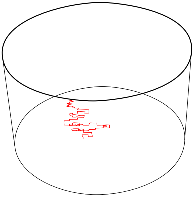

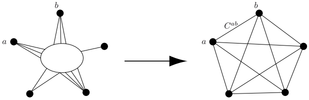

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