Infinite cycles in graphs

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Mathematisches Seminar
Universität Hamburg

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Things that go wrong in infinite graphs

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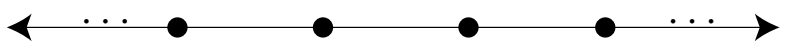
$\Rightarrow$ need more general notions of paths and cycles

Spanning Double-Rays

Classical approach: accept double-rays as infinite cycles



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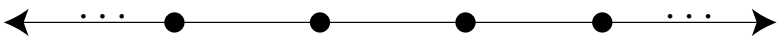

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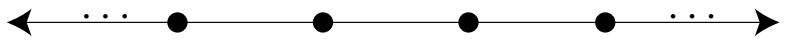

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Theorem (Tutte '56)

Every finite 4-connected planar graph has a Hamilton cycle

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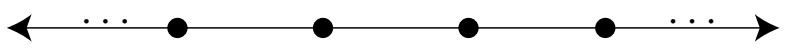

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Theorem (Yu '05)

Every locally finite 4-connected planar graph has a spanning double ray ...

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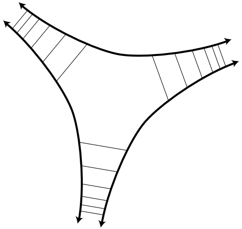
Theorem (Yu '05)

Every locally finite 4-connected planar graph has a spanning double ray ... unless *it is 3-divisible.*

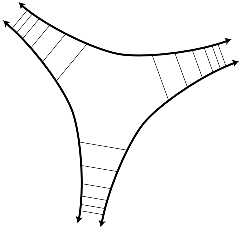
Compactifying by Points at Infinity

A 3-divisible graph



Compactifying by Points at Infinity

A 3-divisible graph
can have no spanning double ray



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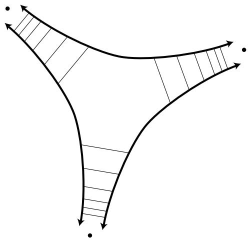

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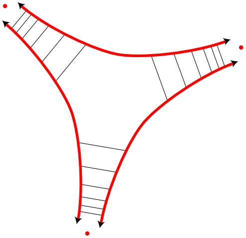

... but a Hamilton cycle?

Ends

end: equivalence class of rays

two rays are equivalent if no finite vertex set separates them

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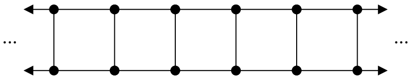

two ends

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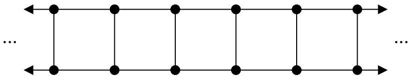

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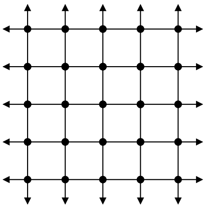

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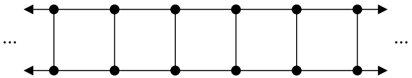

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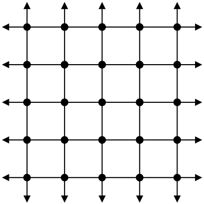

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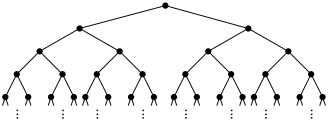
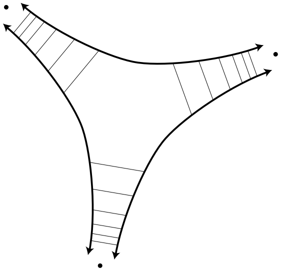


uncountably many ends

The End Compactification



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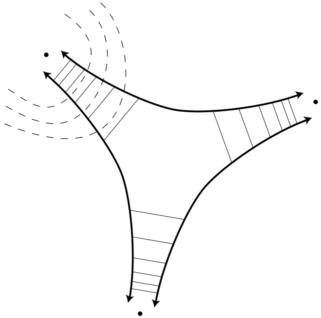

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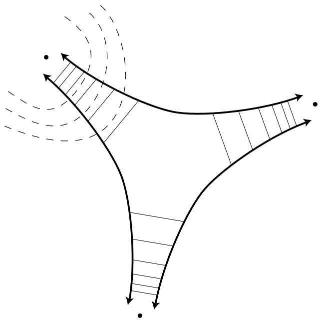
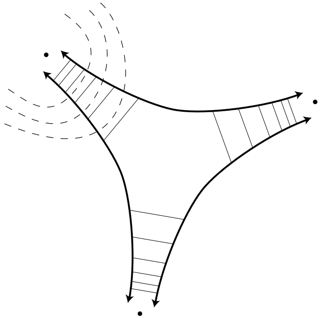


Every ray converges to its end

The End Compactification

$$|G|$$



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Infinite Cycles

Circle:
A homeomorphic image of S^1 in $|G|$.

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a circle containing all vertices

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a circle containing all vertices (and all ends?)

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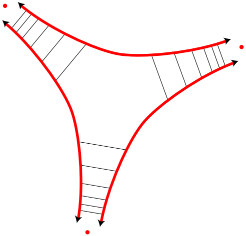

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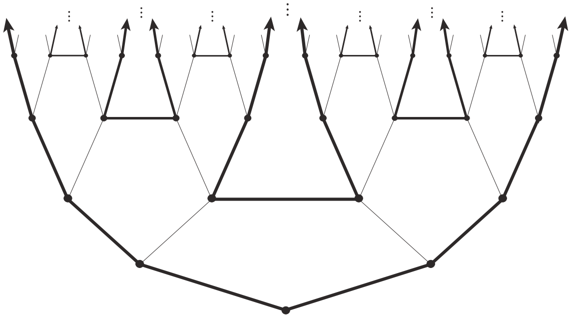

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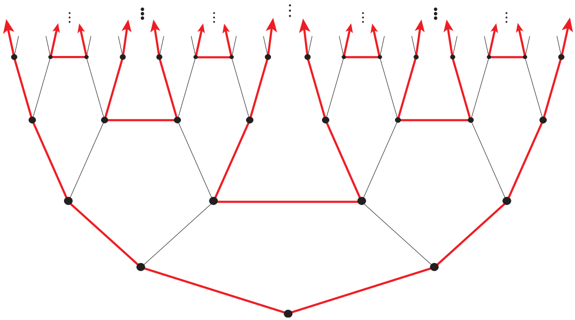

the wild circle of Diestel & Kühn

Fleischner's Theorem

Theorem (Fleischner '74)

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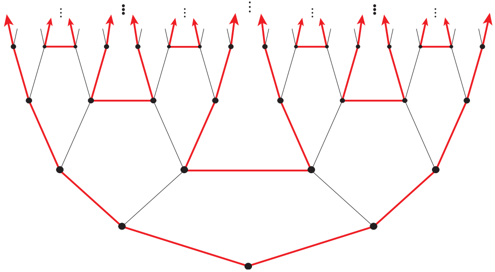
Theorem (Thomassen '78)

The square of a locally finite 2-connected 1-ended graph has a Hamilton circle.

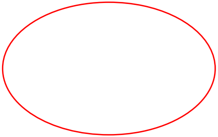
The Theorem

Theorem (G '06)

The square of any locally finite 2-connected graph has a Hamilton circle



Proof?



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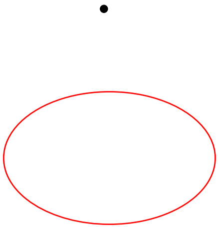

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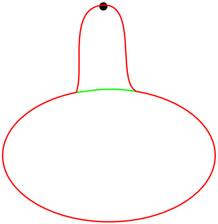

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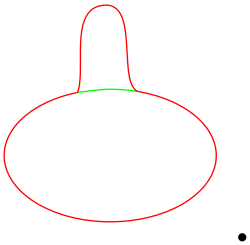

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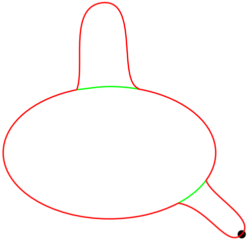

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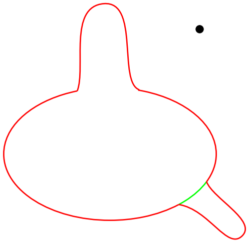

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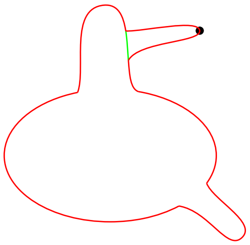

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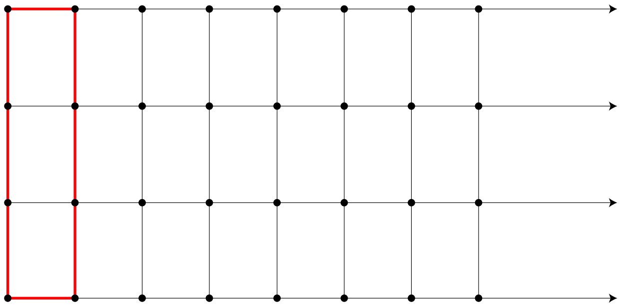

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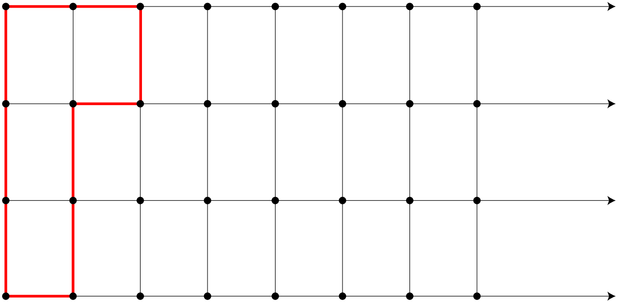

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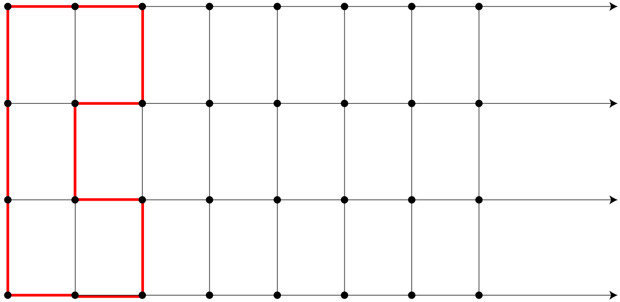

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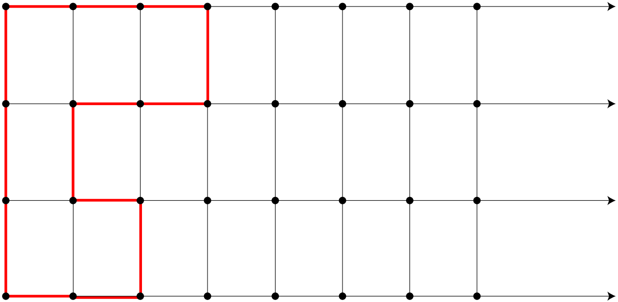

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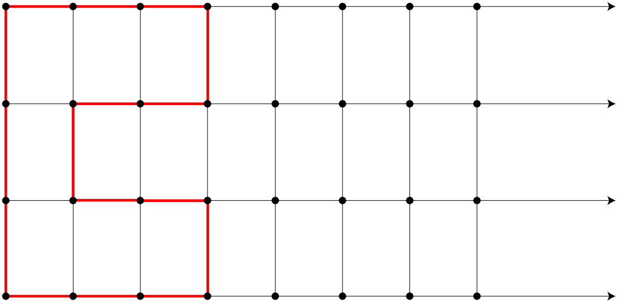

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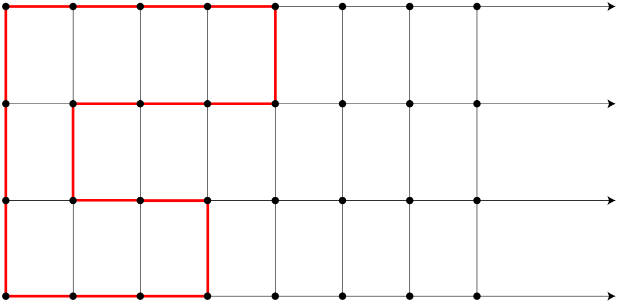

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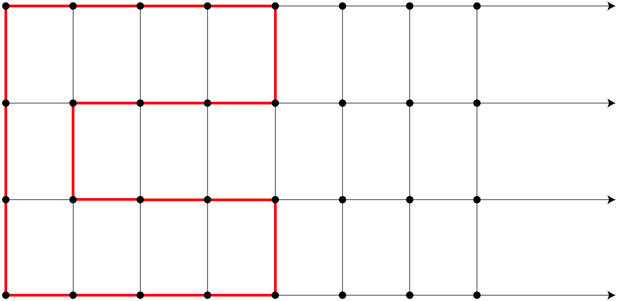

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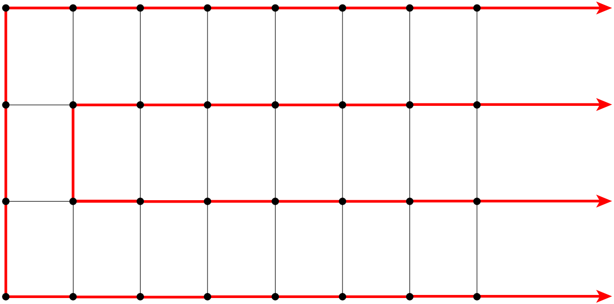

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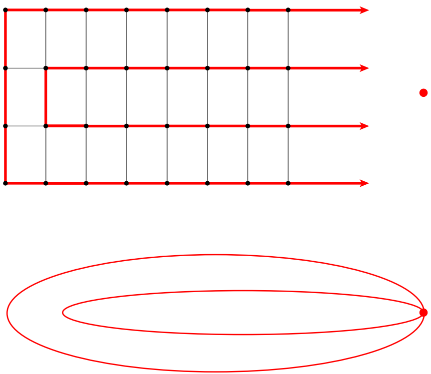
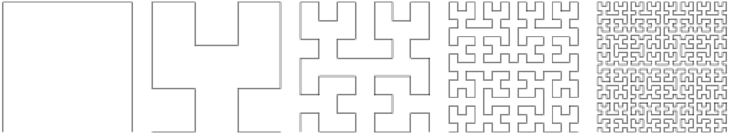


Proof?

Hilbert's space filling curve:



a sequence of injective curves with a non-injective limit

Structure of the Finite Proof

Theorem (G '06)

The square of any locally finite 2-connected graph has a Hamilton circle

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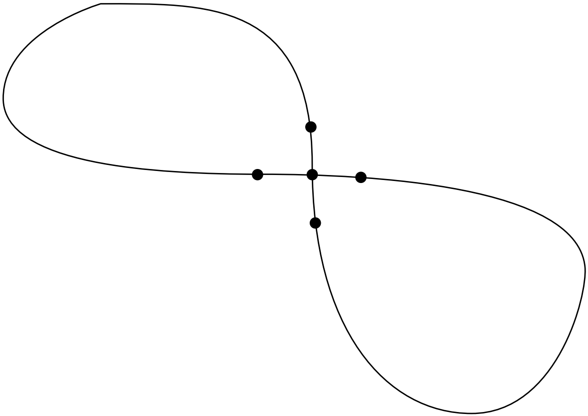

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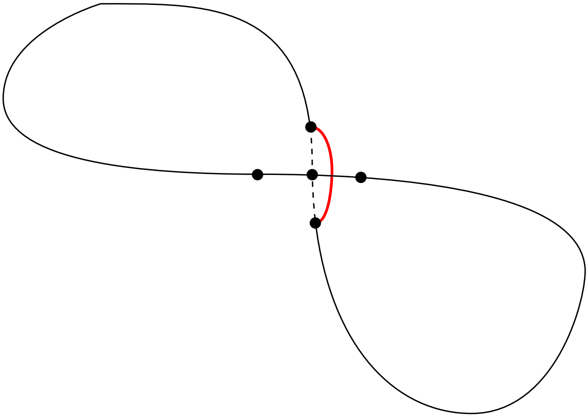

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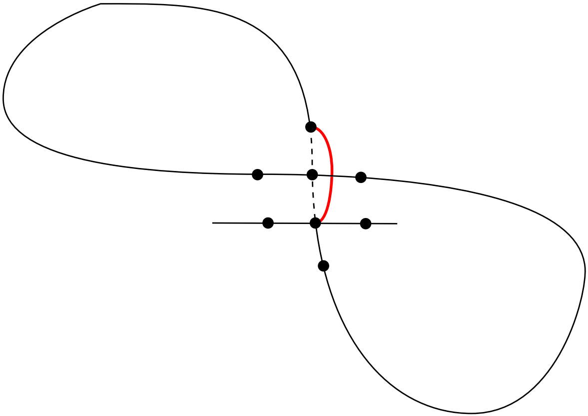

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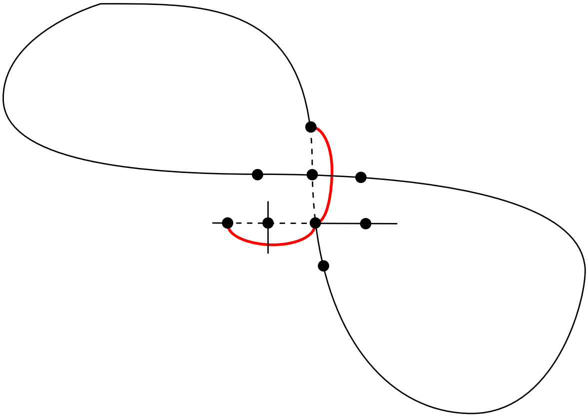

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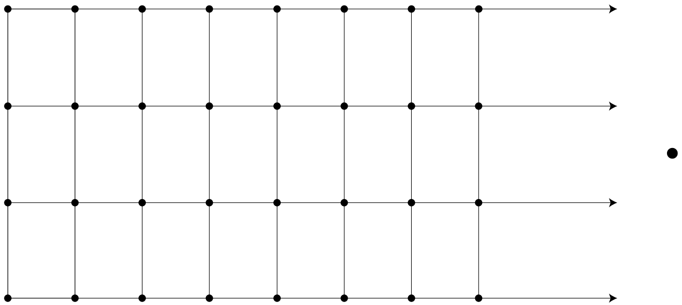

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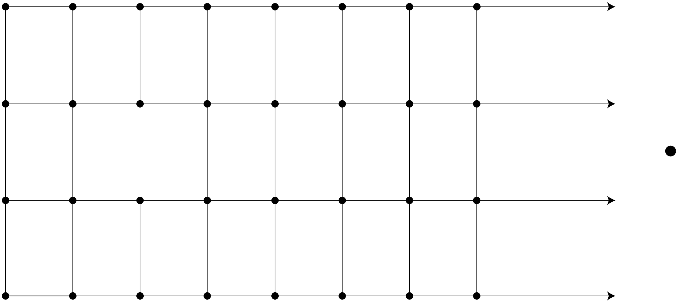

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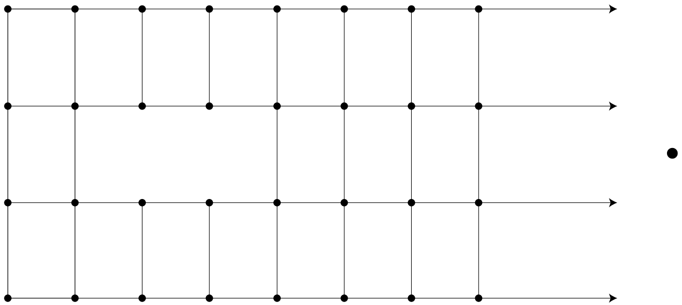

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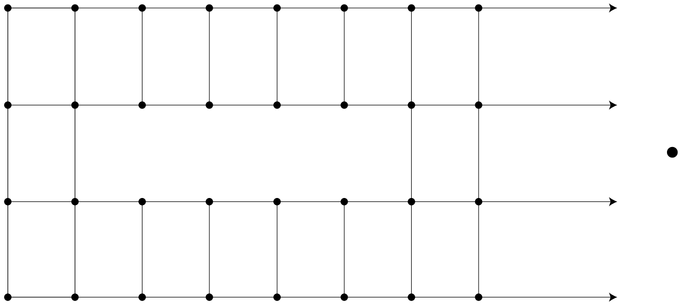

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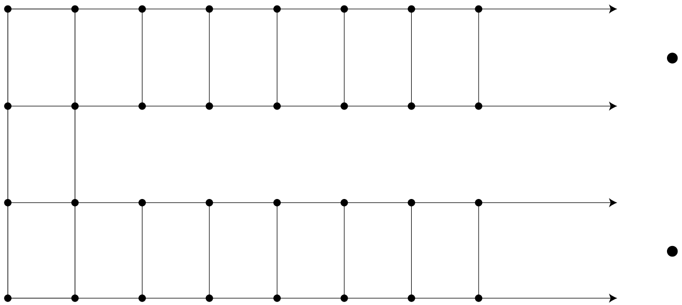

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Hamiltonicity in Cayley graphs

Problem (Rapaport-Strasser '59)

Does every finite connected Cayley graph have a Hamilton cycle?

Hamiltonicity in Cayley graphs

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Problem

Does every connected 1*-ended Cayley graph have a Hamilton circle?*

Hamiltonicity in Cayley graphs

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Problem

Prove that every connected Cayley graph of a finitely generated group Γ has a Hamilton circle unless Γ is the amalgamated product of more than k groups over a subgroup of order k.