Brownian Motion on infinite graphs of finite total length

Agelos Georgakopoulos

Technische Universität Graz

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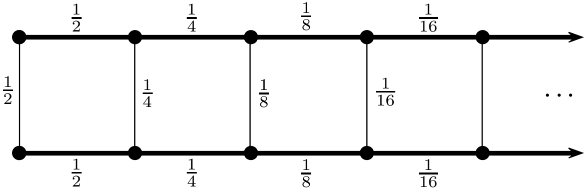

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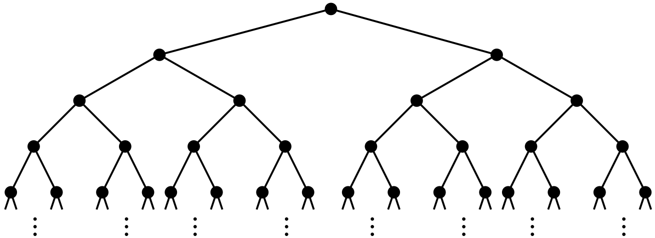

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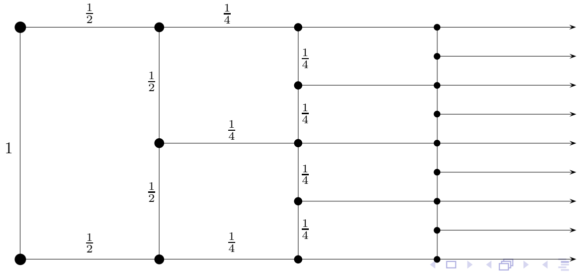

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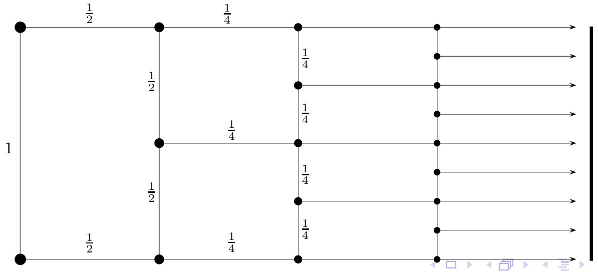

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All above authors "discovered" $|G|_\ell$ independently!

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Strategy: construct brownian motion on $|G|_\ell$ as a limit of brownian motions on finite subgraphs.

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where the f_i are bounded continuous real functions on C

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Theorem (classic)

Let $\Gamma \subseteq \mathcal{M}$. Then $\overline{\Gamma}$ is compact iff for every ϵ there is a function $\omega_\epsilon(\delta)$, with $\omega \to 0$ as $\delta \to 0$, such that $\mu(\{x : w_x(\delta) \leq \omega_\epsilon(\delta) \text{ for all } \delta\}) > 1 - \epsilon/2$ for all $\mu \in \Gamma$,

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$=> \{\mu_n\}_n$ has an accumulation point

Remark: It is known that $\mathcal{M}(X)$ is compact iff X is compact; this would have allowed us to circumvent the above theorem if C were compact, but it isn't (although $|G|_\ell$ is).

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- Transition probabilities coincide with potentials of the corresponding non-elusive electrical current.

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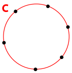

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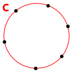

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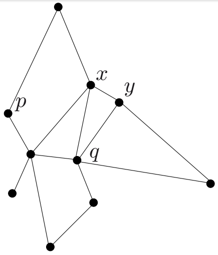
where $v(\vec{e}) := f(\vec{e}) r(e)$ (Ohm's law)

Random Walks & Electrical networks



Random Walks & Electrical networks

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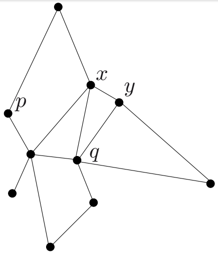

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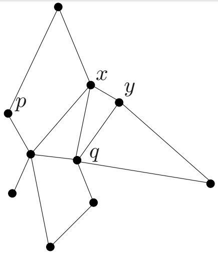

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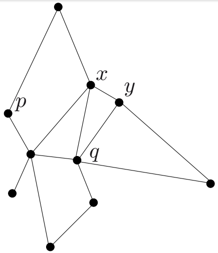

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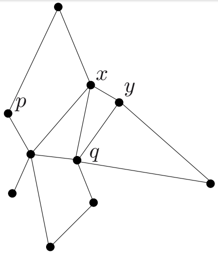

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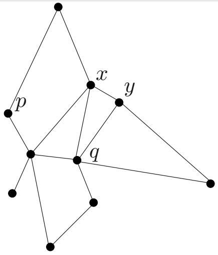

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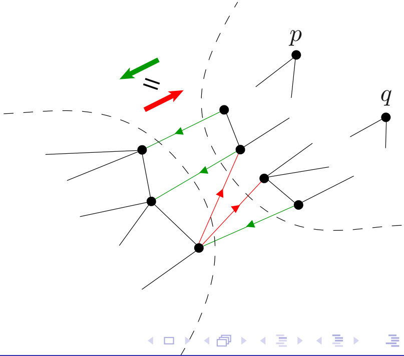

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Non-elusive flow:
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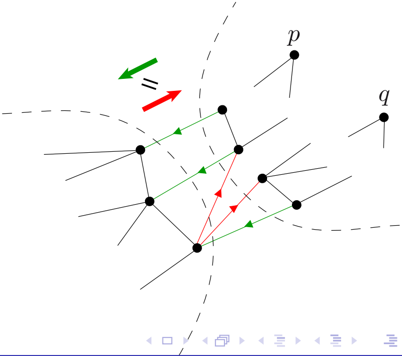

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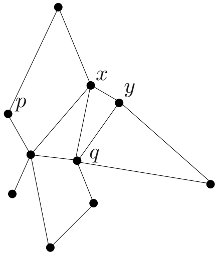

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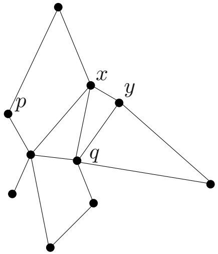

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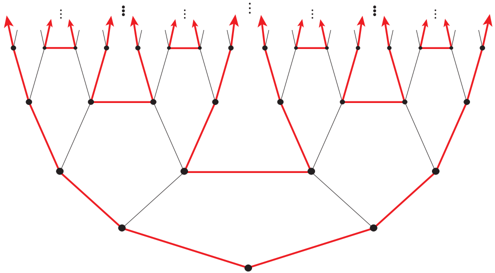

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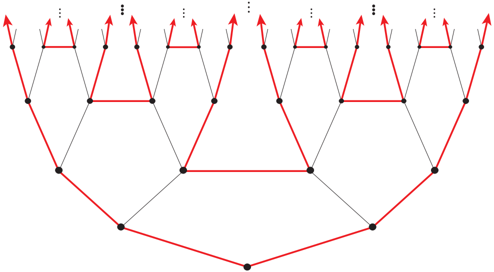

A wild circle i.e. a homeomorphic image of S^1 in $|G|$
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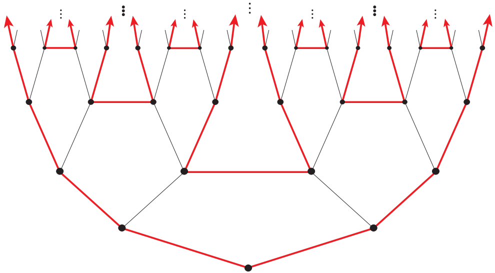

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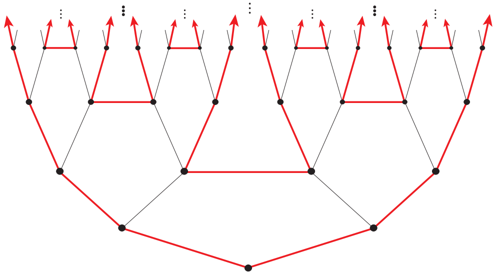
(discovered by Diestel & Kühn)

Contains $\aleph_0$ double-rays aranged like the rational numbers

Wild circles



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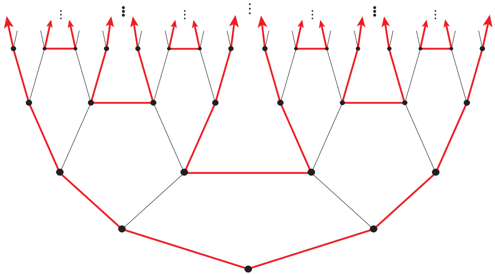
The “gaps” between the double-rays are filled by a Cantor set of ends

Wild circles



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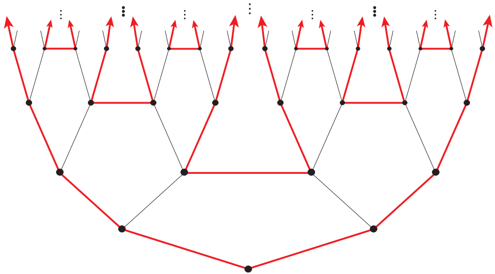
Wild circles



A wild circle i.e. a homeomorphic image of S^1 in $|G|$

More than 30 papers written on wild circles & paths relating to

- Cycle space (Homology)
- Hamilton circles
- Extremal graph theory