Taming the homology of Wild spaces

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(...in other words, looking for a job...)

Strobl, 7.7.11

Overview

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- We are going to *tame* H_1 by removing some 'redundancy'
- ... using experience from infinite graph theory

Example: MacLane's Planarity Criterion

Theorem (MacLane '37)

A finite graph G is planar iff $\mathcal{C}(G)$ *has a simple generating set.*

$\mathcal{C}(G)$: the cycle space of G = $H_1(G)$ (simlicial or singular homology)= $Abel(\pi_1)$
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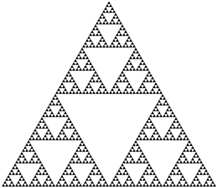

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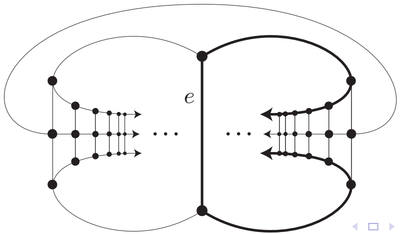

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But using the right homology
(topological cycle space of Diestel & Kühn) ...:

Theorem (Bruhn & Stein '05)

... verbatim generalisation for locally finite G.

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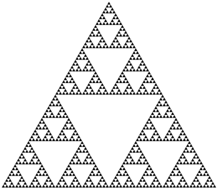

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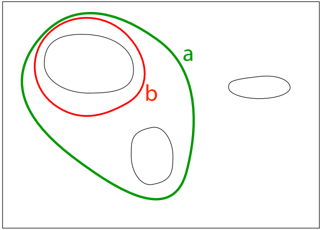

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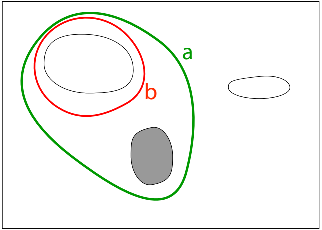

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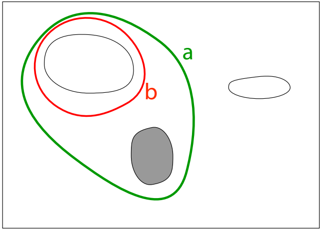

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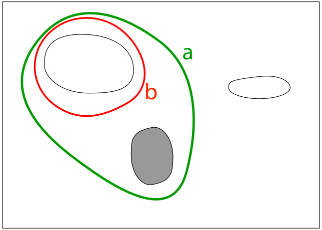

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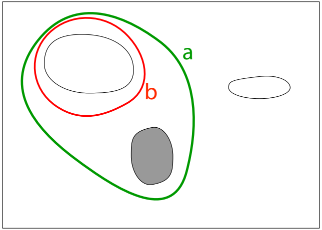

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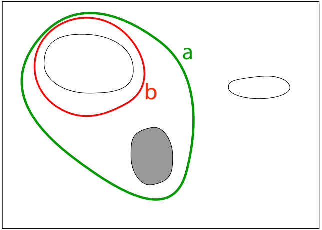

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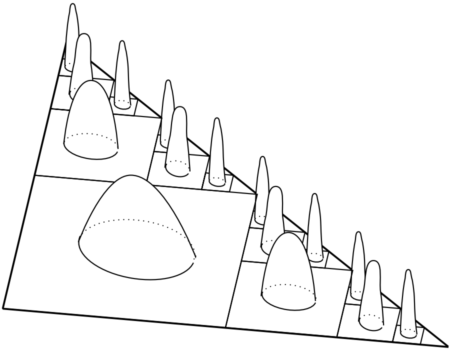
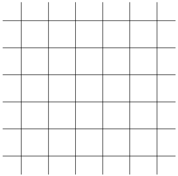
and, if you like, let $\widehat{H_1}(X)$ be its completeion.

Examples



Cycle decompositions



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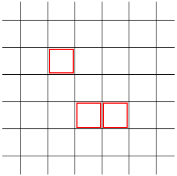

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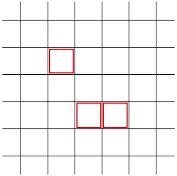

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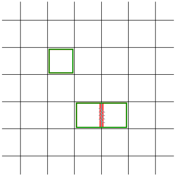

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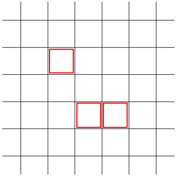

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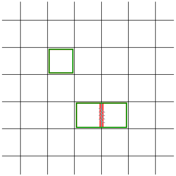

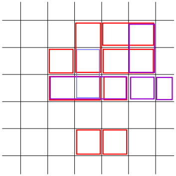

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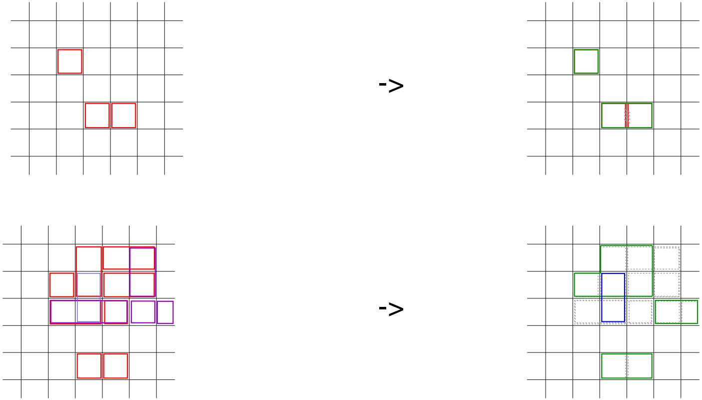

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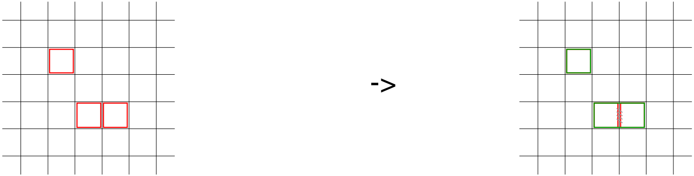

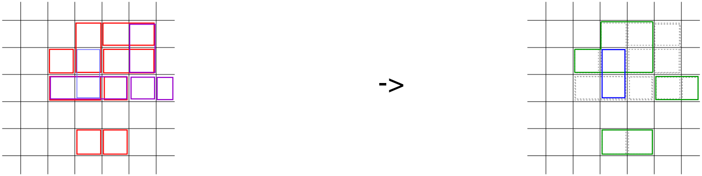

- Can you make a theorem out of this observation?

Cycle decompositions - finite graphs

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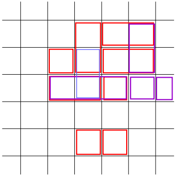
Every element of $\mathcal{C}(G)$ can be written as a union of a set of edge-disjoint cycles.



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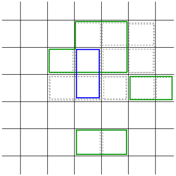

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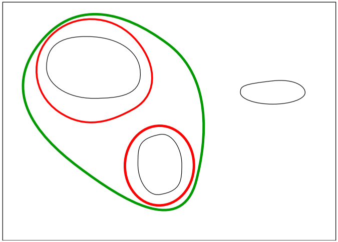
One of many classical theorems recently extended to infinite graphs using our new homology, the topological cycle space $\mathcal{C}(G)$ in an ongoing series of >30 papers by Diestel, Kühn, Bruhn, Stein, G, Sprüssel, Richter, Vella, et. al.

What about more continuous spaces?



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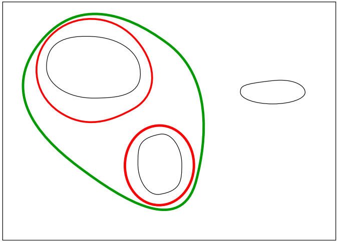

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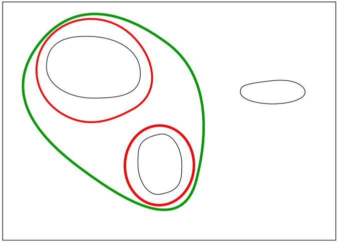

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For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a σ-representative $(z_i)_{i\in\mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.

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We say that $C \in \widehat{H_1}(X)$ splits if there are $A, B \neq 0 \in \widehat{H_1}(X)$ with

$$C = A + B, \text{ and}$$
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Then C is primitive if it doesn't split.

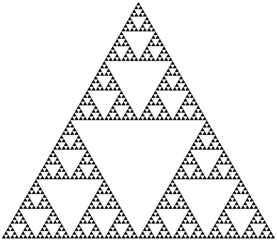
An intermediate result

Let $(\Gamma, +)$ be an abelian metrizable topological group, and suppose a function $\ell : \Gamma \to \mathbb{R}^+$ is given satisfying the following properties

- $\ell(a) = 0$ iff $a = 0$;
- $\ell(a + b) \leq \ell(a) + \ell(b)$ for every $a, b \in \Gamma$;
- if $b = \lim a_i$ then $\ell(b) \leq \liminf \ell(a_i)$;
- Some "isoperimetric inequality" holds: e.g. $d(a, 0) \leq U\ell^2(a)$ for some fixed U and for every $a \in \Gamma$.

Then every element of Γ is a (possibly infinite) sum of primitive elements.

The Conjecture



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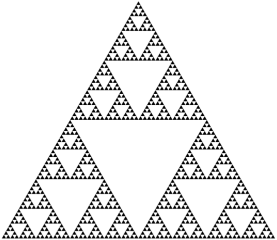

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Conjecture

Let X be a compact, 1–dimensional, locally connected, metrizable space that has no cut point. Then X is planar iff there is a simple set S of loops in X and a metric d inducing the topology of X so that the set $U := \{[\chi] \in \widehat{H_1}(X) \mid \chi \in S\}$ 'spans' $\widehat{H_1}(X)$.

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Theorem (G '06)

If $\sum_{e\in E(G)} \ell(e) < \infty$ *then* $|G|_\ell \approx |G|$. ...and $\widehat{H_1}$ coincides with the topological cycle space and with $\check{H}_1(X)$.

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Problem

Does every compact metrizable space X admit a metric such that $\widehat{H_1}(X) = \check{H}_1(X)$?

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Theorem (Gromov '87)

For every compact metric space X there is a locally finite graph G and $\ell : E \to R_+$ such that the boundary of $|G|_\ell$ is isometric to X.

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All above authors "discovered" $|G|_\ell$ independently!

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- Can you modify $\widehat{H_1}$ to obtain a homology that is invariant under homotopy–equivalence?
- Try to ‘tame’ π_1 by similar methods
- Compute $\widehat{H_1}$ for your favourite space

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- Generalise other graph-theoretical theorems to continua/fractals
- Can you modify $\widehat{H_1}$ to obtain a homology that is invariant under homotopy–equivalence?
- Try to 'tame' π_1 by similar methods
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Sources:

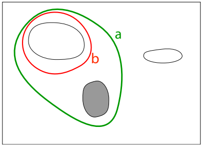
AG: "Cycle decompositions: from graphs to continua", arxiv.org/abs/1003.5115
AG: "Graph topologies induced by edge lengths" http://arxiv.org/abs/0903.1744

These slides are available online

Summary

Theorem (G' 09)

For every compact metric space X and $C \in \widehat{H_1}(X)$, there is a representative $(z_i)_{i\in\mathbb{N}}$ of C that minimizes the length $\sum_i \ell(z_i)$ among all representatives of C.



$$d(a,b) := \inf_{\substack{X \overset{\text{isom}}{\hookrightarrow} X' \\ a \approx b \text{ in } X'}} \text{area}(X' \backslash X). \text{ Let } H_1'(X) := H_1(X)/_{d=0}$$

Conjecture

Let X be a compact, 1–dimensional, locally connected, metrizable space that has no cut point. Then X is planar iff there is a simple set S of loops in X and a metric d inducing the topology of X so that the set $U := \{[\chi] \in \widehat{H_1}(X) \mid \chi \in S\}$ 'spans' $\widehat{H_1}(X)$.