Some results on the interplay between random walks and electrical networks

Agelos Georgakopoulos

University of Warwick

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Electrical Network Reduction

Theorem (G & V. Kaimanovich '11)

Let N be an electrical network and B its set of external nodes. Then there is an equivalent network with vertex set B in which each edge (a, b) has conductance

$$C_{eff}(a, b) = d(a)\mathbb{P}_a(b).$$

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Solution 2: Start $d(b)\hat{u}(b)$ particles at each $b \in B$, kill them upon returning to B, and let

$$u(x) = \frac{\mathbb{E}[\text{\# visits to x}]}{d(x)}$$

Theorem (Kesten '59)

Let Γ be a group generated by a finite set S and let N be a normal subgroup of Γ. Then the following are equivalent:

- $\rho(Cay(\Gamma/N, S)) = \rho(Cay(\Gamma, S))$;
- *N is amenable.*

$\rho(\Gamma) := \lim_n (p_{x,x,2n})^{1/2n}$ is the *spectral radius* of RW on Γ.

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Theorem (T. Lyons '83)

Random Walk on a graph G is transient

<=>

there is a flow of finite **energy** *from some vertex o to infinity.*

Energy $E(i) := \sum_{xy \in E(G)} i(xy)^2$

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For every transient locally finite network N there is a measure $\mathcal{C}$ on $\mathcal{P}^2$ such that for every harmonic function u with boundary function $\widehat{u}$,

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Energy $E(u) := \sum_{x \sim y} (u(x) - u(y))^2$

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$\mathcal{C}$ is equivalent to the square of the Poisson boundary $\mathcal{P}$; thus

$$E(u) = \int_{\mathcal{P}^2} \left(\widehat{u}(\eta) - \widehat{u}(\zeta)\right)^2 \Theta(\eta,\zeta) d\nu^2(\eta,\zeta)$$

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Can a group display > 1 of these modes?

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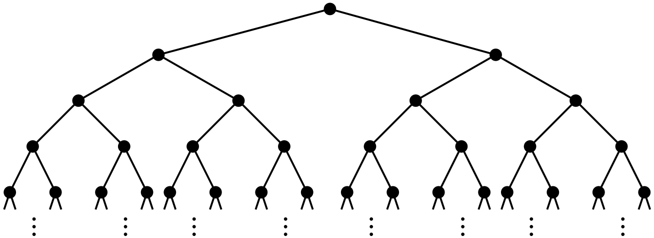

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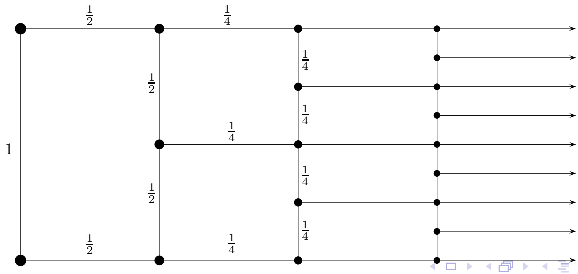

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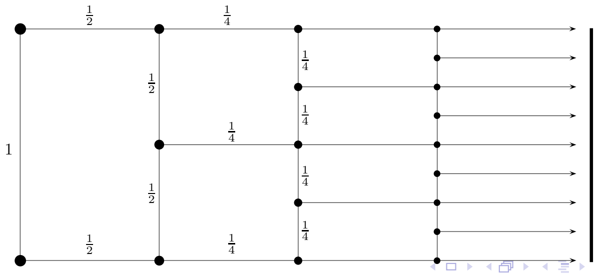

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Theorem (G '06 (easy))

If $\sum_{e \in E(G)} \ell(e) < \infty$ *then* $|G|_\ell \approx |G|$.

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All above authors "discovered" $|G|_\ell$ independently!

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We will construct brownian motion on $|G|_\ell$ as a limit of brownian motions on finite subgraphs.

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where the f_i are bounded continuous real functions on C

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Theorem (classic)

Let $\Gamma \subseteq \mathcal{M}$. Then $\overline{\Gamma}$ is compact iff for every ϵ there is a function $\omega_\epsilon(\delta)$, with $\omega \to 0$ as $\delta \to 0$, such that
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=> $\{\mu_n\}_n$ has an accumulation point

Remark: It is known that $\mathcal{M}(X)$ is compact iff X is compact; this would have allowed us to circumvent the above theorem if C were compact, but it isn't (although $|G|_\ell$ is).

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For every G, ℓ such that $\sum_{e \in E} \ell(e) < \infty$, there is a brownian motion B_ℓ on $|G|_\ell$ with the following properties

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Theorem (G & P. Winkler '11)

The cover time for Brownian motion on a finite graph of total length L is at most $2L^2$.

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Meta-conjecture: (statistical) physics extends to infinite networks of finite total length